A-Level Unit Test: Series and Sequences

Binomial Expansion 1b

Jethwa Maths

1. Find the binomial expansion of $\sqrt[3]{1-x}$ in ascending powers of x up to and including the term in x^3. **(4)**

2a. Find the binomial expansion of $(1 + \frac{1}{4}x)^{-4}$ in ascending powers of x up to and including the term in x^3. **(4)**
b. State the values of x for which the expansion is valid for. **(2)**

3. Expand $(3 - x)^{-3}$ in ascending powers of x up to and including the term x^3 and the set of values of x for which the expansion is valid.
(4)

4. Find the first 4 terms of $\frac{1-x}{\sqrt{1+2x}}$ in ascending powers of x up to and including the term in x^3. State the set of values of x for which each expansion is valid.
(5)

5. Find the series expansion of f(x) in ascending powers of x up to and including the term in x^3 and state the set of value of x for which it is valid.

$$f(x) = \frac{3+5x}{(1+3x)(1+x)^2}$$

(12)

6. The first three terms in the expansion of $(1 + ax)^b$, in ascending powers of x, for $|ax| < 1$, are $1 - 6x + 24x^2$
a. Find the values of the constants a and b **(4)**
b. Find the coefficient of x^3 in the expansion. **(1)**

7. $f(x) = \frac{4}{\sqrt{1+\frac{2}{3}x}}$ $\qquad -\frac{3}{2} < x < \frac{3}{2}$
a. Show that $f(\frac{1}{10}) = \sqrt{15}$ **(2)**
b. Expand f(x) in ascending powers of x up to and including the term in x^2 **(1)**
c. Use your expansion to obtain an approximation for $\sqrt{15}$, giving your answers as an exact, simplified fraction. **(2)**
d. Show that $3\frac{55}{63}$ is more accurate approximation for $\sqrt{15}$ **(2)**

Total marks: 43

Mark Scheme

1.

$(1-x)^{1/3} = 1 + \frac{1}{3}(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-x)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3 \times 2}(-x)^3 + \ldots$	**M1 M1**
$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$	**M1 M1**

2a.

$1 + (-4)(\frac{1}{4}x) + \frac{(-4)(-5)}{2}(\frac{1}{4}x)^2 + \frac{(-4)(-5)(-6)}{3 \times 2}(\frac{1}{4}x)^3 + \ldots$	**M1 M1**
$= 1 - x + \frac{5}{8}x^2 - \frac{5}{16}x^3$	**M1 M1**

2b.

$\left\lvert\frac{1}{4}x\right\rvert < 1,$	**M1**
Therefore, expansion is valid for $\lvert x\rvert < 4$	**M1**

3.

$3^{-3}(1 - \frac{1}{3}x)^{-3} = \frac{1}{27}(1 - \frac{1}{3}x)^{-3}$	**M1**
$= \frac{1}{27}[\,1 + (-3)(-\frac{1}{3}x) + \frac{(-3)(-4)}{2}(-\frac{1}{3}x)^2 + \frac{(-3)(-4)}{3\times2}(-\frac{1}{3}x)^3 + \ldots]$	**M1**
$= \frac{1}{27} + \frac{1}{27}x + \frac{2}{81}x^2 + \frac{10}{729}x^3 + \ldots$	**M1**
$\left\lvert-\frac{1}{3}x\right\rvert < 1$ Therefore expansion is valid for $\lvert x\rvert < 3$	**M1**

4.

$(1-x)(1+2x)^{-0.5} = (1-x)[1 + (-0.5)(2x) + \frac{(-0.5)(-1.5)}{2}(2x)^2 + \frac{(-0.5)(-1.5)(-2.5)}{3\times2}(2x)^3 + \ldots]$	**M1**
$(1-x)(1 - x + 1.5x^2 - 2.5x^3 + \ldots)$	**M1**
$1 - x + 1.5x^2 - 2.5x - 3 - x + x^2 - \frac{3}{2}x^3 + \ldots$	**M1**
$1 - 2x + 2.5x^2 - 4x^3 + \ldots$	**M1**
$\lvert 2x\rvert < 1$ Therefore, expansion is valid for $\lvert x\rvert < \frac{1}{2}$	**M1**

5.

$f(x) = \frac{3+5x}{(1+3x)(1+x)^2} = \frac{A}{1+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$ $3 + 5x = A(1+x)^2 + B(1+3x)(1+x) + C(1+3x)$	**M1**
$x = -\frac{1}{3}, \frac{4}{3} = \frac{4}{9}A$ $A = 3$	**M1**
$x = -1, -2 = -2C$ $C = 1$	**M1**
Comparing coefficients of x^2: $0 = A + 3B$ $B = -1$	**M1**
$\frac{3+5x}{(1+3x)(1+x)^2} = \frac{3}{1+3x} - \frac{1}{1+x} + \frac{1}{(1+x)^2}$	
$\frac{3}{1+3x} = 3(1+3x)^{-1} = 3[1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(3x)^3 + \ldots]$	**M1**
$= 3 - 9x + 27x^2 - 81x^3 + \ldots.$ $\lvert 3x\rvert < 1$ therefore, $\lvert x\rvert < \frac{1}{3}$	**M1**
$\frac{1}{1+x} = 1 + (-1)(x) + \frac{(-1)(-2)}{2}(x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(x)^3 + \ldots$	**M1**

$= 1 - x + x^2 - x^3 + ...$ $\lvert x \rvert < 1$	**M1**
$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 + (-2)(x) + \frac{(-2)(-3)}{2}(x)^2 + \frac{(-2)(-3)(-4)}{3\times 2}(x)^3 + ...$	**M1**
$= 1 - 2x + 3x^2 - 4x^3 + ...$ $\lvert x \rvert < 1$	**M1**
$f(x) = (3 - 9x + 27x^2 - 81x^3 + ...) - (1 - x + x^2 - x^3 + ...) + (1 - 2x + 3x^2 - 4x^3 + ...)$	**M1**
$f(x) = 3 - 10x + 29x^2 - 84x^3 + ...$ As $\lvert x \rvert < \frac{1}{3}$ is the smallest restriction, $\lvert x \rvert < \frac{1}{3}$ for the expansion.	**M1**

6a.

$(1 + ax)^b = 1 + b(ax) + \frac{b(b-1)}{2}(ax)^2 + ...$	**M1**
Therefore, $ab = -6$ and $\frac{1}{2}a^2b(b-1) = 24$	**M1**
$a = -\frac{6}{b}$ Therefore, $\frac{18}{b}(b-1) = 24$	**M1**
$18b - 18 = 24b$ $b = -3$, $a = 2$	**M1**

6b.

$\frac{(-3)(-4)(-5)}{3 \times 2}(2)^3 = -80$	**M1**

7a.

$f(\frac{1}{10}) = \frac{4}{\sqrt{1+\frac{1}{15}}}$	**M1**
$= \frac{4}{\sqrt{\frac{16}{15}}} = \sqrt{15}$	**M1**

7b.

$4(1 + \frac{2}{3}x)^{-0.5} = 4[1 + (-0.5)(\frac{2}{3}x) + \frac{(-0.5)(-1.5)}{2}(\frac{2}{3}x)^2 + ...]$ $= 4 - \frac{4}{3}x + \frac{2}{3}x^2 + ...$	**M1**

7c.

$\sqrt{15} = f(\frac{1}{10}) \approx 4 - \frac{4}{3} \times \frac{1}{10} + \frac{2}{3} \times (\frac{1}{10})^2 + ...$ $= 4 - \frac{2}{15} + \frac{1}{150}$ $= 3\frac{131}{150}$	**M1 M1**

7d.

$\sqrt{15} = 3.87298$ $3\frac{131}{150} = 3.87333$ $3\frac{55}{63} = 3.87301$	**M1**
Therefore, $\sqrt{15} < 3\frac{55}{63} < 3\frac{131}{150}$ Hence, $3\frac{55}{63}$ is a more accurate approximation.	**M1**

A-Level Unit Test: Series and Sequences

Arithmetic Sequences

Jethwa Maths

1. The third and eighth terms of an arithmetic series are 72 and 37 respectively.
a. Find the first term and common difference of the series. **(3)**
b. Find the sum of the first 25 series. **(2)**

2a. Prove that the sum of the first n natural numbers is given by $\frac{1}{2}n(n+1)$. **(2)**
b. Find the sum of the natural numbers from 30 to 100 inclusive. **(2)**

3. The sum, S_n, of the first *n* terms of an arithmetic series is 213 and the sum of the first 10 terms of the series is 295.
a. Find the first term and the common difference of the series. **(4)**
b. Find the number of positive terms in the series. **(2)**
c. Hence, find the maximum value of S_n, the sum of the first *n* terms of the series. **(2)**

4. Three consecutive terms of an arithmetic series are $(7k - 1)$, $(5k + 3)$ and $(4k + 1)$ respectively.
a. Find the value of the constant k
b. Find the smallest positive term in the series. **(2)**
Given also that the series has positive terms, **(3)**
c. Show that the sum of the positive terms of the series is given by $r(4r - 3)$ **(2)**

5a. Evaluate $\sum_{r=1}^{30} 4r$ **(1)**
b. Using your answer to part a, or otherwise, evaluate,
i. $\sum_{r=1}^{30}(4r+1)$ **(2)**
ii. $\sum_{r=1}^{30}(8r-5)$ **(2)**

6. Ahmed begins making annual payments into a savings scheme. He pays in £500 in the first year and the amount he pays in increases by £40 in each subsequent year.
a. Find the amount that Ahmed pays into the scheme in the eighth year. **(1)**
b. Show that during the first *n* years, Ahmed pays in a total amount, in pounds, of $20n(n + 24)$. **(2)**
Carol starts making payments into a similar scheme at the same time as Ahmed. She pays in £400 in the first year, with the amount increasing by £60 each year.
c. By forming and solving a suitable equation, find the number of years of paying into the schemes after which Carol and Ahmed will have paid in the same amount in total.
(4)

7. An arithmetic series has first term *a* and common difference *d*. Given that the sum of the first twenty terms of the series is equal to the sum of the next ten terms of the series, show that the ratio $a : d = 11 : 2$.
(4)

8. A publisher decides to start producing calendars. The publisher sells 2400 calendars during the first year of production and forecasts that the number it will sell in subsequent years will increase by 250 each year.
a. According to this forecast,
i. Find how many calendars the publisher will sell during the sixth year of production, **(1)**
ii. Show that the publisher will sell a total of 35 250 calendars during the first ten years of production. **(2)**
b. If the number of calendars sold in subsequent years increases by *C* each year, instead of by 250, find to the nearest unit the value of *C* such that the publisher would sell a total of 40 000 calendars during the first ten years of production. **(2)**

Total marks: 45

Mark Scheme

1a.

U_3: $a + 2d = 73$ U_8: $a + 7d = 37$	**M1**
Subtracting, $5d = -35$ $d = -7$	**M1**
$a + 2(-7) = 73$ $a = 73 + 13 = 86$	**M1**

1b.

$S_{25} = \frac{25}{2}[172 + (24 \times -7) = 50$	**M1 M1**

2a.

$S_n = 1 + 2 + 3 + \ldots.. + (n-1) + n$ In reverse, $S_n = n + (n-1) + \ldots. + 3 + 2 + 1$	**M1**
$2S_n = n(n+1)$ $S_n = \frac{1}{2}n(n+1)$	**M1**

2b.

$S_{100} - S_{29} = \frac{1}{2} \times 100 \times 101 - \frac{1}{2} \times 29 \times 30$ $= 5050 - 435$	**M1**
$= 4615$	**M1**

3a.

$\frac{6}{2}(2a + 5d) = 213 \rightarrow 2a + 5d = 71$	**M1**
$\frac{10}{2}(2a + 9d) = 295 \rightarrow 2a + 9d = 59$	**M1**
Subtracting $4d = -12$ $d = -3$	**M1**
$2a + 9(-3) = 59$ $a = 43$	**M1**

3b.

$43 - 3(n-1) > 0$	**M1**
$n < \frac{46}{3}$, therefore 15 positive terms	**M1**

3c.

Max S_n when $n = 15$ $S_{15} = \frac{15}{2}[86 + (14 \times -3)$	**M1**
$S_{15} = 330$	**M1**

4a.

$(5k + 3) - (7k - 1) = (4k + 1) - (5k + 3)$	**M1**
$-2k + 4 = -k - 2$ $k = 6$	**M1**

4b.

Given terms = 41, 33, 25 Therefore $d = -8$	**M1**
Smallest positive term = $25 + (3 \times -8) = 1$	**M1**

4c.

Considering the series of positive terms in reverse: $a = 1$, $d = 8$	**M1**
$S_r = \frac{r}{2}[2 + 8(r - 1)] = r(4r - 3)$	**M1**
$S_r = r(4r - 3)$	**M1**

5a.

AP: $a = 4$, $l = 120$, $n = 30$ $S_{30} = \frac{30}{2}(4 + 120) = 1860$	**M1**

5bi.

$\sum_{r=1}^{30}(4r) + 30$	**M1**
$= 1860 + 30 = 1890$	**M1**

5bii.

$2 \times \sum_{r=1}^{30}(4r) - (30 \times 5)$	**M1**
$= (2 \times 1860) - 150$ $= 3570$	**M1**

6a.

$500 + (7 \times 40) = £780$	**M1**

6b.

AP: $a = 500$, $d = 40$ $S_n = \frac{n}{2}[1000 + 40(n - 1)]$	**M1**
$= 20n(n + 24)$	**M1**

6c.

AP: $a = 400$, $d = 60$	**M1**
$S_n = \frac{n}{2}[800 + 60(n - 1)] = 10n(3n + 37)$	**M1**
Therefore, $20n(n + 24) = 10n(3n + 37)$ $2n + 48 = 3n + 37$	**M1**
$n = 11$ Therefore, 11 years.	**M1**

7.

$S_{20} = \frac{20}{2}(2a + 19d) = 20a + 190d$ $S_{30} = \frac{30}{2}(2a + 29d) = 20a + 435d$	**M1**
$S_{30} - S_{20} = 10a + 245d$	**M1**
$20a + 190d = 10a + 245$ $10a = 55$	**M1**
$2a = 11d$ $a : d = 11 : 2$	**M1**

8ai.

$2400 + (5 \times 250) = 3650$	**M1**

8aii.

$a = 2400$, $d = 250$ $S_{10} = \frac{10}{2}[4800 + (9 \times 250)$	**M1**
$= 35\ 250$	**M1**

8b.

$a = 2400$, $d = C$ $\frac{10}{2}[4800 + (9 \times C) = 40000$	**M1**
$C = \frac{3200}{9} = 356$ (to the nearest unit)	**M1**

A-Level Unit Test: Series and Sequences

Geometric Sequences

Jethwa Maths

1. The first and fourth terms of a geometric series are 2 and 54 respectively.
a. Find the common ratio of the series **(2)**
b. Find the ninth term of the series **(1)**

2. The first and third terms of a geometric series are 6 and 24 respectively.
a. Find the two possible values for the common ratio of the series. **(2)**
Given also that the common ratio of the series is positive
b. Find the sum of the first 15 terms of the series. **(2)**

3. All the terms of a geometric series are positive. The sum of the first and second terms of the series is 10.8 and the sum of the third and fourth terms of the series is 43.2
a. Find the first term and common ratio of the series **(4)**
b. Find the sum of the first 16 terms of the series **(1)**

4. The common ratio of a geometric series is 0.55 and the sum to infinity of the series is 40
a. Find the first term of the series **(3)**
b. Find the smallest value of n for which the nth term of the series is less than 0.001 **(2)**

5. A car was purchased for £18000 on 1st January. On 1st January each following year, the value of the car is 80% of its value on 1st January in the previous year
a. Show that the value of the car exactly three years after it was purchased Is £9216
The value of the car falls below £1000 for the first time *n* years after it was purchased **(2)**
b. Find the value of *n* **(3)**
An insurance company has a scheme to cover the maintenance of the car. The cost is £200 for the first year, and for every following year the cost increases by 12% so that for the 3rd year the cost of the scheme is £250.88
c. Find the cost of the scheme for the 5th year, giving your answer to the nearest penny **(2)**
d. Find the total cost of the insurance scheme for the first 15 years **(3)**

6. A trading company made a profit of £50 000 in 2006 (year 1). A model for future trading predicts that profits will increase year by year in a geometric sequence with common ratio *r*, *r*, > 1. The model therefore predicts that in 2007 (Year 2) a profit of £50 000*r* will be made
a. Write down an expression for the predicted profit in Year *n* **(1)**
The model predicts that in year n, the profit made will exceed £200 00
b. Show that $n > \frac{\log 4}{\log r} + 1$ **(3)**
Using the model with $r = 1.09$
c. Find the year in which the profit made will first exceed £200 000 **(2)**
d. Find the total of the profits that will be made by the company over the 10 years from 2006 to 2015 inclusive, giving your answer to the nearest £10,000 **(3)**

7. A geometric sequence is $a + ar + ar^2 + \dots$
a. Prove that the sum of the first n terms of this series is given by: $S_n = \frac{a(1-r^n)}{1-r}$ **(4)**
b. Find $\sum_{k=1}^{10} 100(2^k)$ **(3)**
c. Find the sum to infinity of the geometric series $\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \cdots$ **(3)**
d. State the condition for an infinite geometric series with common ratio r to be convergent **(1)**

Total marks: 47

Mark Scheme

1a.

$a = 2$, $ar^3 = 54$ $r^3 = 54 \div 2 = 27$	**M1**
$r = 3$	**M1**

1b.

$U_9 = 2 \times 3^8 = 13122$	**M1**

2a.

$a = 6$ $ar^2 = 25$ $r^2 = 24 \div 6 = 4$	**M1**
$r = \pm 2$	**M1**

2b.

As $r > 0$, $r = 2$ $r = \sqrt[3]{64} = 4$	**M1**
$a \times 4 = 0.5$ $a = 0.125$	**M1**

3a.

$a + ar = a(1 + r) = 10.8$	**M1**
$ar^2 + ar^3 = ar^2(1 + r) = 43.2$ $r^2 = 43.2 \div 10.8 = 4$	**M1**
As all r terms are positive, $r = 2$	**M1**
Therefore a = $10.8 \div 3 = 3.6$	**M1**

3b.

$S_{16} = \frac{3.6(2^{16}-1)}{2-1} = 235\,926$	**M1**

4a.

$\frac{a}{1-0.55} = 40$	**M1**
$a = 0.45 \times 40 = 18$	**M1**

4b.

$18 \times (0.55)^{n-1} < 0.001$ $(n - 1)\log 0.55 < \log 0.0000556$	**M1**
$n > \frac{\log 0.0000556}{\log 0.55} + 1$ $n > 17.4$	**M1**
Therefore smallest $n = 18$	**M1**

5a.

Value after three years = $18000 \times 0.8 \times 0.8 \times 0.8 = 18000 \times 0.8^3$ = £9216	**M1**

5b.

$18000 \times 0.8^n < 1000$ $0.8^n < 0.0\dot{5}$	**M1**
$\log 0.8^n < \log 0.0\dot{5}$ $n \log 0.8 < \log 0.0\dot{5}$	**M1**

$n > 12.952$ $n = 13$ $\text{Log}_a N$ is negative if $0 < N < 1$	**M1**

5c.

Cost in 5th year = 200×1.12^4	**M1**
= £314.703 = £314.70 (to the nearest pence)	**M1**

5d.

Total cost for 15 years = $S_{15} = \frac{200(1.12^{15}-1)}{1.12-1}$	**M1** **M1**
= £7455.942 = £745.94 (to the nearest pence)	**M1**

6a.

Profit in year n = $50000\text{r}^{\text{n-1}}$	**M1**

6b.

$50000r^{\text{n-1}} > 2000\,000$ $r^{\text{n-1}} > 4$	**M1**
$\log \text{r}^{\text{n -1}} > \log 4$ $(n-1)\log r > \log 4$	**M1**
$n - 1 > \frac{\log 4}{\log r}$ $n > \frac{\log 4}{\log r} + 1$	**M1**

6c.

$n > \frac{\log 4}{\log r} + 1$ $r = 1.09$ $n > \frac{\log 4}{\log 1.09} + 1$ $n > 17.086$ Therefore $n = 18$	**M1**
1st year = 2007 + 17 = 2023	**M1**

6d.

Profit in year n = $50000\text{r}^{\text{n-1}}$ Total profit in the first 10 years = $S_{10} = \frac{50000[1.09^{10}-1]}{1.09-1}$	**M1** **M1**
S_{10} = £759646.48 = £760 000 (to nearest £10 000)	**M1**

7a.

Let $S_n = a + ar + ar^2 + ar^3 + \ldots. + ar^{n-2} + ar^{n-1}$ $rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots. + ar^{n-1} + ar^n$	**M1**
$S_n - rS_n = a = ar^n$ $S_n(1-r) = a(1-r^n)$	**M1**
$S_n(1-r) = a(1-r^n)$	**M1**
$S_n = \frac{a(1-r^n)}{1-r}$	**M1**

7b.

$\sum_{k=1}^{10} 100(2^k) = 100(2^1) + 100(2^2) + 100(2^3) + \ldots + 100(2^{10})$	**M1**
$= \frac{100(2)[1-2^{10}]}{1-2}$	**M1**
= 204600	**M1**

7c.

$\frac{5}{6}+\frac{5}{18}+\frac{5}{54}+\cdots=\frac{5}{6}+\frac{5}{6}(\frac{1}{3})^2+\frac{5}{6}\left(\frac{1}{3}\right)^2+\frac{5}{6}\left(\frac{1}{3}\right)^3+\ldots$	**M1**
$=\frac{\frac{5}{6}}{1-\frac{1}{3}}$	**M1**
$=\frac{5}{4}$	**M1**

7d.

The common ratio must satisfy: $-1 < r < 1$	**M1**

A-Level Unit Test: Series and Sequences

Recurrence Relations

Jethwa Maths

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1. For the sequence $U_n = 4U_{n-1} - k, n > 1, U_1 = k$, find expression for U_2 and U_3 in terms of the constant *k*. **(2)**

2. For the sequence $U_{n+1} = \frac{U^n}{k}, n > 1, U_1 = 4$, find expression for U_2 and U_3 in terms of the constant *k*. **(2)**

3. For the sequence $U_{n+1} = \sqrt[3]{61k^3 + U_n{}^3}, n > 0, U_1 = k\sqrt[3]{3}$, find expression for U_2 and U_3 in terms of the constant *k*. **(2)**

4. The terms of a sequence $U_1, U_2, U_3 \ldots$ are given by $U_n = 3(U_{n-1} - k)$, n > 1 where k is constant. Given that $U_1 = -4$

a. Find expressions for U_2 and U_3 in terms of *k*. **(2)**

Given also that $U_3 = 7U_2 + 3$, find

b. The value of *k*, **(2)**

c. The value of U_4. **(2)**

5. The *n*th term of a sequence, U_n , is given by

$$U_n = k^n - 3.$$

Given that $U_1 + U_2 = 0$,

a. Find the two possible values of the constant *k*. **(3)**

b. For each value of *k* found in part a, find the corresponding value of U_5. **(2)**

6. A sequence $U_1, U_2, U_3, \ldots$ is defined by

$$U_{n+1} = \frac{1}{U_n} \qquad U_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} U_r$ **(4)**

Total marks: 21

Mark Scheme

1.

$U_1 = k$ $U_2 = 4k - k = 3k$	**M1**
$U_3 = 4(3k) - k = 16 + 15k$	**M1**

2.

$U_1 = 4$ $U_2 = \frac{4}{k}$	**M1**
$U_3 = \frac{4}{k} \div k = \frac{4}{k^2}$	**M1**

3.

$U_1 = k\sqrt[3]{3}$ $U_2 = \sqrt[3]{61k^3 + (k\sqrt[3]{3})^3} = \sqrt[3]{61k^3 + 3k^3} = \sqrt[3]{64k^3} = 4k$	**M1**
$U_3 = \sqrt[3]{61k^3 + (4\text{k})^3} = \sqrt[3]{61k^3 + 64k^3} = \sqrt[3]{125k^3} = 5k$	**M1**

4a.

$U_1 = -4$ $U_2 = 3(-4 - k) = -12 - 3k$	**M1**
$U_3 = 3[(-12 - 3k) - k] = -36 - 12k$	**M1**

4b.

$-36 - 12k = 7(-12 - 3k) + 3$ $9k = -45$	**M1**
$k = -5$	**M1**

4c.

$U_3 = -36 - 12(-5) = 24$	**M1**
Therefore, $U_4 = 3(24 + 5) = 87$	**M1**

5a.

$U_1 = k - 3$ $U_2 = k^2 - 3$	**M1**
$U_1 + U_2 = 0$ $k - 3 + (k^2 - 3) = 0$ $k^2 + k - 6 = 0$	**M1**
$(k + 3)(k - 2) = 0$ $k = -3 \text{ or } 2$	**M1**

5b.

$k = -3$ $U_5 = (-3)^5 - 3 = -242 - 3 = -246$	**M1**
$k = 2$ $U_5 = (2)^5 - 3 = 32 - 3 = 29$	**M1**

6.

$U_1 = \frac{2}{3}$ $U_2 = \frac{3}{2}$ $U_3 = \frac{2}{3}$	**M1**

$U_4 = \frac{3}{2}$	
Sequence is periodic with order 2. Therefore, $\sum_{r=1}^{100} U_r = (50 \text{ x } U_1) + (50 \text{ x } U_2)$	**M1**
$\sum_{r=1}^{100} U_r = \left(50 \times \frac{2}{3}\right) + \left(50 \times \frac{3}{2}\right)$	**M1**
$= \frac{325}{3}$	**M1**