A-Level Unit Test: Iteration

1. The diagram shows the curve, $y = f(x)$, where f is the function defined for all real values of x by

$$f(x) = 3 + 4e^{-x}$$

a. State the range of $f(x)$. **(1)**

b. Find an expression for $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$ **(4)**

c. The straight line $y = x$ meets the curve $y = f(x)$ at the point P. By using an iterative process based on the equation $y = f(x)$, with a starting value of 3, find the coordinates of the point P. Show all your working and give each coordinate correct to 3 decimal places. **(2)**

d. How is the point P related to the curves $y = f(x)$ and $y = f^{-1}(x)$ **(1)**

2. $g(x) = e^{x-1} + x - 6$

a. Show that the equation $g(x) = 0$ can be written as $x = \ln(6 - x) + 1$, $x < 6$ **(2)**

The root of $g(x) = 0$ is α

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \quad x_0 = 2$$

is used to find an approximate value of α

b. Calculate the values of x_1, x_2 and x_3 to 4 decimal places. **(3)**

c. By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. **(3)**

3. $f(x) = 4\operatorname{cosec} x - 4x + 1$, where x is in radians.

a. Show that there is a root α of $f(x) = 0$, in the interval [1.2, 1.3] **(2)**

b. Show that the equation $f(x) = 0$ can be written in the form $x = \frac{1}{\sin x} + \frac{1}{4}$ **(2)**

c. Use the iterative formula

$$x_{n+1} = \frac{1}{\sin x_n} + \frac{1}{4}, \quad x0 = 1.25$$

to calculate the values of x_1, x_2 and x_3 giving your answers to 4 decimal places. **(3)**

d. By considering the change in sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.291$ correct to 3 decimal places. **(2)**

Total marks: 25

Mark Scheme

1a.

As $x \to \infty$, $\frac{1}{e^x} \to 0$ Therefore, $f(x) \to 3$ Range: $f(x) > 3$	**M1**

1b.

$f(x) = 3 + \frac{4}{e^x}$ Let $x = 3 + \frac{4}{e^y}$	**M1**
$xe^y = 3e^y + 4$ $e^y(x - 3) = 4$ $e^y = \frac{4}{x-3}$ $y = \ln \frac{4}{x-3}$	**M1**
Domain: $x > 3$	**M1**
Range: $f^{-1}(x)$ can be any real number.	**M1**

1c.

$x_1 = 3.1991\ldots = 3.199$ (3 d.p)	**M1**
$x_2 = 3.1631\ldots = 3.163$ (3 d.p) $x_3 = 3.1691\ldots = 3.169$ (3 d.p) $x_4 = 3.1681\ldots = 3.168$ (3 d.p) $x_5 = 3.1683\ldots = 3.168$ (3 d.p)	**M1**
$x_6 = 3.1682\ldots = 3.168$ (3 d.p)	**M1**

1d.

P is at the point of intersection of $y = f(x)$ and $y = f^{-1}(x)$	**M1**

2a.

When $g(x) = 0$ $e^{x-1} + x - 6 = 0$ $e^{x-1} = (6 - x)$	**M1**
$x - 1 = \ln(6 - x)$ $x = \ln(6 - x) + 1$	**M1**

2b.

$x_1 = 2.3863$	**M1**
$x_2 = 2.2847$	**M1**
$x_3 = 2.3125$	**M1**

2c.

$2.3065 < \alpha < 2.3075$	**M1**
$f(2.3065) = -0.000275\ldots.$ $f(2.3075) = 0.004419\ldots.$	**M1**
As there is a change in sign the root is in the interval.	**M1**

3a.

when $f(x) = 0$ $f(1.2) = \frac{3}{\sin(1.2)} - 4(1.2) + 1 = 0.49166\ldots.$ $f(1.3) = \frac{3}{\sin(1.3)} - 4(1.3) + 1 = -0.0487$	**M1**
Since there is a change in sign and the curve is continuous over the interval [1.2, 1.3] then there is a root in the interval.	**M1**

3b.

$\frac{4}{\sin x} + 1 = 4x$ $4 + \sin x = 4x \sin x$	**M1**
$\frac{4}{4\sin x} + \frac{\sin x}{4 \sin x} = x$ $x = \frac{1}{\sin x} + \frac{1}{4}$	**M1**

3c.

$x_0 = 1.15$ $x_1 = 1.3038$ (4 d.p)	**M1**
$x_2 = 1.2867$ (4 d.p)	**M1**
$x_3 = 1.2917$ (4 d.p)	**M1**

3d.

f(1.2905) = $\frac{4}{\sin(1.2905)} - 4(1.2905) + 1 = 0.000456$ f(1.2915) = $\frac{4}{\sin(1.2915)} - 4(1.2915) + 1 = -0.00475$	**M1**
As there is a change in sign, α is accurate to 1.29.	**M1**

A-Level Unit Test: Newton-Raphson

1. $f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5, x > 0$

a. Find $f'(x)$ **(2)**

The equation $f(x) = 0$ has a root α in the interval [4.5, 5.5].

b. Using $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 significant figures. **(4)**

2. $f(x) = x^3 - \frac{7}{x} + 2, x > 0$

a. Show that $f(x) = 0$ has a root α between 1.4 and 1.5 **(2)**

b. Starting with the interval [1.4, 1.5], use interval bisection twice to find an interval of width 0.025 that contains α **(3)**

c. Taking 1.45 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x) = x^3 - \frac{7}{x} + 2$ to obtain a second approximation to α, giving your answer to 3 decimal places. **(5)**

3. $f(x) = x^3 - \frac{5}{2x^{-\frac{3}{2}}} + 2x - 3, x > 0$

a. Show that the equation $f(x) = 0$ has a root α in the interval [1.1, 1.5] **(2)**

b. $f'(x)$ **(2)**

c. Using $x0 = 1.1$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. **(3)**

Total marks: 23

<u>Mark Scheme</u>

1a.

f'(x) = $x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$	**M1 M1**

1b.

$x_1 = x_0 - \frac{f(x_n)}{f'(x_n)}$	**M1**
$x_1 = 5 - \frac{2(5)^{\frac{1}{2}}+(5)^{-\frac{1}{2}}-5}{(5)^{-\frac{1}{2}}-\frac{1}{2}(5)^{-\frac{3}{2}}}$ = 5.2003…. = 5.20 (to 3 significant figures)	**M1**

2a.

f(1.4) = -0.256 f(1.5) = 0.783…	**M1**
As there is a change in sign, there must be a root [1.4, 1.5] as the curve is continuous.	**M1**

2b.

x_1 = 1.45 f(1.45) = 0.22103 Therefore, root lies in [1.4, 1.45] $x_2 = \frac{1.4+1.45}{2} = 1.425$	**M1**
f(1.425) = -0.0186	**M1**
Therefore root α lies in [1.425, 1.45]	**M1**

2c.

x_1 = 1.45 f(x) = $x^3 - 7x^{-1} + 2$ f'(x) = $3x^2 + 7x^{-2}$	**M1**
$x_2 = 1.45 - \frac{(1.45)^3 - \frac{7}{1.45} + 2}{3(1.45)^2 + \frac{7}{1.45^2}}$ = 1.42706….	**M1**
x_2 = 1.427 (to 3 d.p)	**M1**

3a.

f(1.1) = -1.63596… f(1.5) = 2.01417…	**M1**
As there is a change in sign, there must be a root [1.1, 1.5] as the curve is continuous.	**M1**

3b.

f'(x) = $3x^2 + \frac{15}{4}x^{-\frac{5}{2}} + 2$	**M1 M1**

3c.

f'(1.1) = 8.58494…	**M1**
$x_1 = 1.1 + \frac{1.63596}{8.58494}$ = 1.2905…	**M1**
x_1 = 1.292 (to 3 d.p)	**M1**