Numerical Methods

Pt. 1: Iteration

A-Level

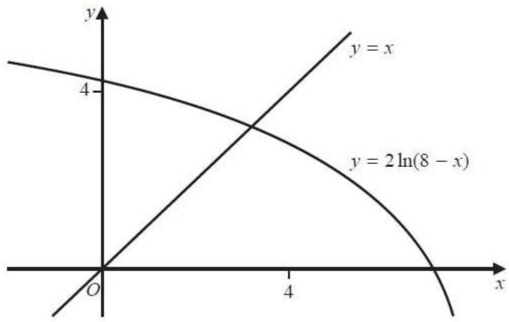
Pt.1 : Iteration

Pt. 2: Newton-Raphson Method

Jethwa Maths

1. The curve with equation $y = 2 \ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

a. Show that $3 < \alpha < 4$. **(2)**



The figure shows the graph of $y = 2 \ln(8 - x)$ and the graph of $y = x$. A students uses the iteration formula $x_{n+1} = 2 \ln(8 - x_n)$ in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

b. Determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer. **(2)**

2. Figure 2 shows a sketch of part of the curve with equation,

$$y = 2\cos\left(\frac{1}{2}x^2\right) + x^3 - 3x - 2$$

The curve crosses the x-acis at the point Q and has a minimum turning point at R.

a. Show that the x-coordinate of Q lies between 2.1 and 2.2. **(2)**

b. Show that the x-coordinate of R is a solution of the equation. **(4)**

$$x = \sqrt{1 + \frac{2}{3}x\sin\left(\frac{1}{2}x^2\right)}$$

Using the iterative formula:

$$x_{n+1} = \sqrt{1 + \frac{2}{3}x_n\sin\left(\frac{1}{2}{x_n}^2\right)}$$

c. Find the values of x_1 and x_2 to 3 decimal places. **(2)**

3. $f(x) = 25x^2e^{2x} - 16$

a. Using calculus, find the exact coordinates of the turning points on the curve with equation $y = f(x)$. **(5)**

b. Show that the equation $f(x) = 0$, can be written as $x = = \pm\frac{4}{5}e^{-x}$ **(1)**

The equation $f(x) = 0$ has a root α, where $\alpha = 0.5$ to 1 decimal place.

c. Starting with $x_0 = 0.5$, use the iteration formula $x_{n+1} = \frac{4}{5}e^{-x_n}$ to calculate the values of x_1, x_2 and x_3 giving your answers to 3 decimal places. **(3)**

d. Give an accurate estimate for α to 2 decimal places, and justify your answer. **(2)**

<u>Mark Scheme</u>

1a.

At the point of intersection, 2 ln(8 – x) = x 2 ln (8 – x) – x = 0	**M1**
f(3) = 0.2188 f(4) = -1.2274 There is a root α in the interval [3, 4] since there is a change in sign and the curve is continuous over the interval.	**M1**

1b.

Demonstration of cobweb graph	**M1**
The cobweb graph sprials inwards towards α, threfeore the iteration formula can be used as an approximation for α.	**M1**

2a.

When x = 2.1, y = -0.2240…. When x = 2.2, y = 0.5464…	**M1**
Therefore x-coordainte of Q lies between 2.1 and 2.2 since there is a change of sign and the curve is continuous.	**M1**

2b.

Attempt to differentiate $y = 2\cos(\frac{1}{2}x^2) + x^3 - 3x - 2$	**M1**
$\frac{dy}{dx} = 2\left[-\sin\left(\frac{1}{2}x^2\right)\right](x) + 3x^2 - 3$	**M1**
At R, $\frac{dy}{dx} = 0$ $3x^2 = 3 + 2x(\sin\frac{1}{2}x^2)$	**M1**
$x^2 = 1 + \frac{2}{3}x\sin(\frac{1}{2}x^2)$ $x = \sqrt{1 + \frac{2}{3}x\sin(\frac{1}{2}x^2)}$	**M1**

2c.

x_1 = 1.28383… = 1.284 (3 d.p)	**M1**
x_2 = 1.2760… = 1.2760 (3 d.p)	**M1**

3a.

$y = 25x^2e^{2x} - 16$ $\frac{dy}{dx} = 25x^2(2e^{2x}) + (e^{2x})(50x)$	**M1**
$\frac{dy}{dx} = 50x^2e^{2x} + 50xe^{2x} = 50xe^{2x}(x + 1)$	**M1**
At turning point, $\frac{dy}{dx} = 0$, $50xe^{2x}(x + 1) = 0$ $x = 0$ $x = -1$	**M1**
When $x = 0$, $y = 25(0)e^0 - 16 = -16$	**M1**
When $x = -1$, $y = 25(-1)^2e^{-2} - 16 = 25e^{-2} - 16$	**M1**
Turning points: (0, -16), (-1, $25e^{-2} - 16$)	

3b.

When $f(x) = 0$, $25x^2e^{2x} - 16 = 0$ $x^2 = \frac{16}{25e^{2x}}$ $x = \pm\frac{4}{5}e^{-x}$	**M1**

3c.

$x_0 = 0.5$ $x_1 = 0.4852\ldots = 0.485$ (to 3d.p)	**M1**
$x_2 = 0.4924\ldots = 0.492$ (to 3d.p)	**M1**
$x_3 = 0.4889\ldots = 0.489$ (to 3d.p)	**M1**

3d.

$\alpha = 0.49$ $f(0.485) = -0.4872\ldots$ $f(0.495) = 0.4854\ldots.$	**M1**
As there is a change in sign and the function is continuous, $\alpha = 0.49$ (to 2 d.p)	**M1**

<u>Mark Scheme</u>

1.

	M1
	M1

Numerical Methods
Pt. 2: Newton-Raphson Method

A-Level
Pt.1 : Iteration
Pt. 2: Newton-Raphson Method

1. The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

a. Find the Newton-Raphson formula for this equations. **(3)**

Using your formula from part a with $x_1 = 1$,

b. Find the values of x_2 and x_3. **(2)**

c. Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$. **(1)**

2. $f(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, x \geq 0$
The root α of the equation $f(x) = 0$ lies in the interval [1.6, 1.8].

a. Use linear interpolationonce on the interval [1.6, 1.8] to find an approximation to α. Give your answer to 3 decimal places. **(4)**

b. Differentiate $f(x)$ to find $f'(x)$. **(2)**

c. Taking 1.7 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4)**

3a. Show that $f(x) = x^4 + x - 1$ has a real root α in the interval [0.5, 1.0]. **(2)**

b. Starting with the interval [0.5, 1.0], use interval bisection twice to find an interval of width 0.125 which contains α. **(3)**

c. Taking 0.75 as a first approximation, apply the Newton-Raphson process twice to $f(x)$ to obtain an approximate value of α. Give your answer to 3 decimal places. **(4)**

Mark Scheme

1a.

let $f(x) = 2x^3 + x^2 - 1$ $f'(x) = 6x^2 + 2x$	**M1**
$x_{n+1} = x_n - \frac{2x_n^3 + x_n^2 - 1}{6x_n^2 + 2x_n}$ $= \frac{x_n(6x_n^2 + 2x_n) - (2x_n^3 + x_n^2 - 1)}{6x_n^2 + 2x_n}$	**M1**
$= \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$	**M1**

1b.

$x_2 = \frac{4(1)^3 + (1)^2 + 1}{6(1)^2 + 2(1)} = \frac{6}{8} = \frac{3}{4}$	**M1**
$x_3 = \frac{4(\frac{3}{4})^3 + (\frac{3}{4})^2 + 1}{6(\frac{3}{4})^2 + 2(\frac{3}{4})} = \frac{\frac{13}{4}}{\frac{39}{8}} = \frac{2}{3}$	**M1**

1c.

$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$	**M1**
$x_3 = \frac{4(\frac{3}{4})^3 + (\frac{3}{4})^2 + 1}{6(\frac{3}{4})^2 + 2(\frac{3}{4})} = \frac{\frac{13}{4}}{\frac{39}{8}} = \frac{2}{3}$	**M1**

2a.

f(1.6) = -1.295430 …. f(1.8) = 0.540186…	**M1**
$\frac{\alpha - 1.6}{1.8 - \alpha} = \frac{1.295430}{0.540186}$	**M1**
$\alpha - 1.6 = (2.39811….)(1.8 - \alpha)$	**M1**
$\alpha = 1.741$	**M1**

2b.

$f'(x) = 10x - 6x^{\frac{1}{2}}$	**M1 M1**

2c.

$x_1 = 1.7$ $x_2 = 1.7 - \frac{f(1.7)}{f'(1.7)}$	**M1**
$x_2 = 1.7 - \frac{5(1.7)^2 - 4(1.7)^{\frac{3}{2}} - 6}{10(1.7) - 6(1.7)^{\frac{1}{2}}}$	**M1 M1**
$x_2 = 1.7 - \frac{-0.4161527...}{9.176957...}$	**M1**
$x_2 = 1.74534… = 1.745$ (3.dp)	**M1**

3a.

f(0.5) = -0.4375 f(1.0) = 1	**M1**
A root lies in the interval since there is a change in sign and f(x) is continuous over this interval.	**M1**

3b.

0.75 - 0.125 = 0.625	**M1**
f(0.75) = 0.06640	**M1**
f(0.625) = -0.2224	**M1**

3c.

$f(x) = x^4 + x - 1$ $f'(x) = 4x^3 + 1$	**M1**
$x_1 = 0.75$ $x_2 = 0.75 - \frac{(0.75)^4 + (0.75) - 1}{4(0.75)^3 + 1}$	**M1**
$x_2 = 0.7525290\ldots = 0.725$ (to 3 d.p)	**M1**
$x_3 = 0.72449\ldots = 0.724$ (to 3 d.p)	**M1**