Integration
Pt. 4: Exponentials, Reverse Chain Rule and Trig

Jethwa Maths

1. Integrate with respect to x, $\frac{2}{5x} - \frac{3e^t}{7}$ **(2)**

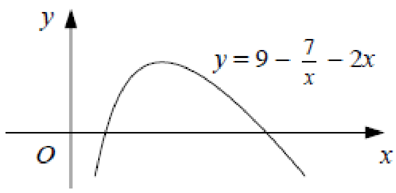
2. Find, $\int \frac{3x+1}{x}\, dx$ **(2)**

3. The curve y = f(x) passes through the point (1, -3). Given that f'(x) = $\frac{(2x-1)^2}{x}$, find an expression for f(x) **(4)**

4. Evalulate, $\int_{-2}^{-1} \frac{6x+1}{3x} dx$ **(4)**

5. Find the exact value of $\int_{\ln 2}^{\ln 4} (7 - \ e^x)\, dx$ **(3)**

6. The diagram shows the curve with equation y = 9 – $\frac{7}{x} - 2x, x > 0$.



a. Find the coordintes of the points where the curve crosses the x-axis. **(3)**
b. Show that the area of the region bound by the cruve and the x-axis is $11\frac{1}{4} - 7\ln\frac{7}{2}$ **(4)**

7. Integrate with resprect to x, $\frac{1}{2(5-3x)^3}$ **(3)**

8. Find $\int 5e^{7-3x}\, dx$ **(2)**

9. Evaluate $\int_4^7 (\frac{x-4}{2})^3\, dx$ **(4)**

10. y = $(3x – 5)^3$. Find the exact area of the region enclosed by the coordaintes (2, 1) and (3, 64) **(4)**

11a. Show that $\int \frac{3x+5}{(x+1)(x+3)}$ in partial fractions. **(4)**
b. Hence, find $\int \frac{3x+5}{(x+1)(x+3)}\, dx$ **(2)**

12. Integrate with respect to x, $\frac{14-x}{x^2+2x-8}$ **(7)**

13. Integrate $\frac{x^2-x+1}{x^2-3x-10}$ with respect to x **(7)**

14. Find the exact value of $\int_1^2 \frac{9}{2x^2-7x-4} dx$ **(8)**

15. Find the integral of the curve y = $\frac{4}{sin^2 x}$ **(2)**

16. Evaluate $\int_0^{\frac{\pi}{2}} 2\sec\frac{1}{2}\tan\frac{1}{2}x\,dx$ **(2)**

17a. Use the identity for cos (A + B) to express $\cos^2 A$ in terms of $\cos 2A$. **(4)**
b. Find $\int cos^2x\,dx$ **(2)**

18. Find $\int cos^4x\,dx$ **(5)**

19. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} sec^2x\,cosec^2x\,dx$ **(4)**

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Mark Scheme

1.

$\frac{2}{5} ln\|t\| - \frac{3}{7}e^t + c$	**M1 M1**

2.

$\frac{3x+1}{x} = \frac{3x}{x} + \frac{1}{x} = 3 + \frac{1}{x}$	**M1**
$\int 3 + \frac{1}{x} = 3x + \ln\|x\| + c$	**M1**

3.

$f'(x) = \frac{4x^2-4x+1}{x} = 4x - 4 + \frac{1}{x}$	**M1**
$f(x) = \int 4x - 4 + \frac{1}{x}\ dx = 2x^2 - 4x + \ln\|x\| + c$	**M1**
At (1, -3) → $-3 = 2 - 4 + 0 + c$	**M1**
$c = -1$	**M1**
$f(x) = 2x^2 - 4x + \ln\|x\| - 1$	

4.

$\int_{-2}^{-1} \frac{6x+1}{3x} dx = \int_{-2}^{-1} 2 + \frac{1}{3x} dx$	**M1**
$= [2x + \frac{1}{3} ln\|x\|]_{-2}^{-1}$	**M1**
$= (-2 + 0) - (-4 + \frac{1}{3}\ln 2)$	**M1**
$= 2 - \frac{1}{3}\ln 2$	**M1**

5.

$\int_{\ln 2}^{\ln 4} (7 - e^x)\ dx = [7x - e^x]_{\ln 2}^{\ln 4}$	**M1**
$= (7 \ln 4 - 4) - (7 \ln 2 - 2)$	**M1**
$= 7 \ln 2 - 2$	**M1**

6a.

$9 - \frac{7}{x} - 2x = 0$	**M1**
$2x^2 - 9x + 7 = 0$ $(2x - 7)(x - 1) = 0$	**M1**
$x = 1$ $x = \frac{7}{2}$	**M1**
Therefore (1, 0) and $(\frac{7}{2}, 0)$	

6b.

$\int_1^{\frac{7}{2}} 9 - \frac{7}{x} - 2x\ dx$	**M1**
$= [9x - 7\ ln\|x\| - x^2]_0^{\frac{7}{2}}$	**M1**
$= (\frac{63}{2} - 7\ln\frac{7}{2} - \frac{49}{4}) - (9 - 0 - 1)$	**M1**
$= 11\frac{1}{4} - 7\ln\frac{7}{2}$	**M1**

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7.

$= \int \frac{1}{2}(5-3x)^{-3}\,dx$	**M1**
$= -\frac{1}{3} \times \frac{1}{-4}\ (5-3x)^{-2} + c$	**M1**
$= \frac{1}{12(5-3x)^2} + c$	**M1**

8.

$= -\frac{5}{3}e^{7-2x} + c$	**M1**

9.

$= \frac{1}{8}\int_4^7 (x-4)^3\,dx$	**M1**
$= \frac{1}{8}[\frac{1}{4}(x-4)^4]_4^7$	**M1**
$= \frac{1}{32}(81-0)$	**M1**
$= 2\frac{17}{32}$	**M1**

10.

$\int_2^3 (3x-5)^3 dx$	**M1**
$[\frac{1}{3} \times \frac{1}{4}(3x-5)^4]_2^3$	**M1**
$\frac{1}{12}(256-1)$	**M1**
$= 21\frac{1}{4}$	**M1**

11a.

$\frac{3x+5}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$	**M1**
$3x + 5 = A(x + 3) + B(x + 1)$	**M1**
Let $x = -1$, $2 = 2A$ $A = 1$	**M1**
Let $x = -3$, $-4 = -2B$ $B = 2$	**M1**
$\frac{3x+5}{(x+1)(x+3)} = \frac{1}{x+1} + \frac{2}{x+3}$	

11b.

$\int \frac{1}{x+1} + \frac{2}{x+3}\ \mathrm{d}x$	**M1**
$= \ln\|x+1\| + 2\ln\|x+3\| + c$	**M1**

12.

$\frac{14-x}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2}$	**M1**
14 – x = A(x – 2) + B(x + 4)	**M1**
Let x = -4, 18 = -6A A = -3	**M1**
Let x = 2, 12 = 6B B = 2	**M1**
$\frac{14-x}{x^2+2x-8} = \frac{-3}{x+4} + \frac{2}{x-2}$	**M1**
$\int \frac{-3}{x+4} + \frac{2}{x-2}\ \mathrm{d}x$	**M1**
= $2\ln\lvert x-2\rvert - 3\ln\lvert x+4\rvert$ + c	**M1**

13.

$\frac{x^2-x+1}{x^2-3x-10} = A + \frac{B}{x-5} + \frac{C}{x+2}$	**M1**
$x^2 - x + 1$ = A(x – 5)(x + 2) + B(x – 2) + c(x – 5)	**M1**
Let x = 5, 21 = 7B B = 3	**M1**
Let x = -2, 7 = -7C C = -1	**M1**
Coefficients of x^2, A = 1	**M1**
$\int \frac{x^2-x+1}{x^2-3x-10}dx = \int 1 + \frac{3}{x-5} - \frac{1}{x+2}\ \mathrm{d}x$	**M1**
= x + $3\ln\lvert x-5\rvert - \ln\lvert x+2\rvert$ + c	**M1**

14.

$\frac{9}{2x^2-7x-4} = \frac{A}{2x+1} + \frac{B}{x-4}$	**M1**
9 = A(x – 4) + B(2x + 1)	**M1**
Let x = 4, 9 = 9B B = 1	**M1**
Let x = $-\frac{1}{2}$ 9 = $-\frac{9}{2}A$ A = -2	**M1**
$\int_1^2 \frac{9}{2x^2-7x-4}dx = \int_1^2 \frac{-2}{2x+1} + \frac{1}{x-4}dx$	**M1**
= $[\ln\lvert x-4\rvert - \ln\lvert 2x+1\rvert]_1^2$	**M1**
= (ln 2 – ln 5) = (ln 3 – ln 3)	**M1**
= ln 2 – ln 5	**M1**

15.

$\int \frac{4}{sin^2x}\ dx = \int 4cosec^2x\ dx$	**M1**
= - 4cot x + c	**M1**

16.

$\int_0^{\frac{\pi}{2}} 2\sec\frac{1}{2}\tan\frac{1}{2}x\ dx$ = $[4\sec\frac{1}{2}x]_0^{\frac{\pi}{2}}$	**M1**
= $4\sqrt{2}$ – 4 = 4($\sqrt{2}$ – 1)	**M1**

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17a.

$\cos(A + B) = \cos A \cos B - \sin A \sin B$	**M1**
Let $B = A$, $\cos 2A = \cos^2 A - \sin^2 A$	**M1**
$= \cos^2 A - (1 - \cos^2 A)$ $= 2\cos^2 A - 1$	**M1**
$= \frac{1}{2}(1 + \cos 2A)$	**M1**

17b.

$\int cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2A) \, dx$	**M1**
$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$	**M1**

18.

$\int cos^4 x \, dx = \int [\frac{1}{2}(1 + cos 2x)]^2 \, dx$	**M1**
$= \frac{1}{4}\int (1 + 2\cos 2x + cos^2 2x) \, dx$ $= \frac{1}{4}\int [1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)] \, dx$	**M1**
$= \frac{1}{8}\int [3 + 4\cos 2x + \cos 4x] \, dx$	**M1**
$= \frac{1}{8}(3x + 2\sin 2x + \frac{1}{4}\sin 4x) + c$	**M1**
$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$	**M1**

19.

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} sec^2 x \, cosec^2 x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{cos^2 x} \times \frac{1}{sin^2 x} \, dx$	**M1**
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{(\frac{1}{2}sin 2x)^2} \, dx$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4 \, cosec^2 2x \, dx$	**M1**
$= [-2\cot 2x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	**M1**
$= \frac{2}{\sqrt{3}} - (-\frac{2}{\sqrt{3}})$ $= \frac{4}{3}\sqrt{3}$	**M1**

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Integration

Pt. 5: Integration by Susbstitution

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1. Use the substitution $u = x^2 + 1$, to find the value of $\int 2x(x^2 - 1)^3\, dx$ **(4)**

2. Use the substitution $u = \cos 2x$, $\int \sin 2x\, cos^3 2x\, dx$ **(4)**

3. Using the substitution $u = x^2 - 3$, find the exact value of $\int_1^2 x(x^2 - 3)^3 dx$ **(5)**

4. Using the susbstitution $u = 1 + e^{2x}$, $\int_0^1 e^{2x}(1 + e^{2x})^3\, dx$ **(5)**

5. Using the substitution, $u = 2 - x^2$ evalulate $\int_0^1 xe^{2-x^2}\, dx$ **(5)**

6. Using the substitution $x = \sin u$, find $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\, dx$ **(5)**

7. Using the susbstitution $x = 3 \tan u$, find the exact value of $\int_0^3 \frac{x^2}{x^2+9} dx$ **(6)**

Mark Scheme

1.

$u = x^2 + 1$ $\frac{du}{dx} = 2x$	**M1**
$\int 2x(x^2 - 1)^3\ dx\ = \int 2x\ u^3\ dx$ $= \int 2x\ u^3 \frac{du}{2x}$ $= \int u^3 du$	**M1**
$= \frac{1}{4}u^4 + c$	**M1**
$= \frac{1}{4}(x^2 + 1)^4 + \text{c}$	**M1**

2.

$u = \cos 2x$ $\frac{du}{dx} = \ -2\sin 2x$	**M1**
$\int \sin 2x\ cos^3 2x\ dx\ =\ \int \sin 2x\ u^3\ \frac{du}{-\sin 2x}$ $= \int -\frac{1}{2}u^3\ du$	**M1**
$= -\frac{1}{8}u^4 + \text{c}$	**M1**
$= -\frac{1}{8}cos^4 2x + \text{c}$	**M1**

3.

$u = x^2 - 3$ $\frac{du}{dx} = 2x$	**M1**
$x = 1, u = -2$ $x = 2, u = 1$	**M1**
$\int_1^2 x(x^2 - 3)^3 dx = \int_{-2}^{1} \frac{1}{2}u^3\ du$	**M1**
$= [\frac{1}{8}u^4]_{-2}^{1}$	**M1**
$\frac{1}{8}(1 - 16) = -\frac{15}{8}$	**M1**

4.

$u = 1 + e^{2x}$ $\frac{du}{dx} = 2e^{2x}$	**M1**
$x = 0, u = 2$ $x = 1, u = 1 + e^2$	**M1**
$\int_0^1 e^{2x}(1 + \ e^{2x})^3\ dx = \int_2^{1+e^2} \frac{1}{2}u^3\ du$	**M1**
$= [\frac{1}{8}u^4]_{2}^{1+e^2}$	**M1**
$= \frac{1}{8}[(1 + \ e^2)^4 - 16]$ $= \frac{1}{8}(1 + \ e^2)^4 - 2$	**M1**

5.

$u = 2 - x^2$ $\frac{du}{dx} = -2x$	**M1**
$x = 1, u = 1$ $x = 0, u = 2$	**M1**
$\int_0^1 xe^{2-x^2}\ dx = \int_1^2 \frac{1}{2} e^u\ du$	**M1**
$[\frac{1}{2}e^u]_1^2$	**M1**
$= \frac{1}{2}(e^2 - e)$ $= \frac{1}{2}e(e-1)$	**M1**

6.

$x = \sin u$ $\frac{du}{dx} = \cos u$	**M1**
$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx = \int \frac{1}{cos^3 u} \times \cos u\ \ du$ $= \int sec^2 u\ \ du$	**M1**
$= \tan u + c$	**M1**
$= \frac{\sin u}{\cos u} + c$	**M1**
$= \frac{x}{\sqrt{1-x^2}} + c$	**M1**

7.

$x = 3 \tan u$ $\frac{dx}{du} = 3\ sec^2 u$	**M1**
$x = 0, u = 0$ $x = 3, u = \frac{\pi}{4}$	**M1**
$\int_0^3 \frac{x^2}{x^2+9} dx = \int_0^{\frac{\pi}{4}} \frac{9tan^2 u}{9sec^2 u} \times 3sec^2 u\ \ du$	**M1**
$= 3\int_0^{\frac{\pi}{4}} tan^2 u\ \ du$ $= 3\int_0^{\frac{\pi}{4}} (sec^2 u - 1)\ \ du$	**M1**
$= 3[\tan u - u]_0^{\frac{\pi}{4}}$	**M1**
$= 3[(1 - \frac{\pi}{4}) - (0)]$ $= \frac{3}{4}(4 - \pi)$	**M1**

Jethwa Maths

Integration

Pt. 6: Integration by Parts

A-Level

Pt. 4: Exponentials, Reverse Chain Rule and Trig
Pt. 5: Integration by Substitution
Pt. 6: Integration by Parts
Pt. 7: Parametric Integration
Pt. 8: Differential Equations

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1. Use integration by parts to show that $\int x\cos x\,dx = x\sin x + \cos x + c$ **(3)**

2. Use integration by parts to find $\int x sec^2 x\,dx$ **(4)**

3. Find the exact value of $\int_0^2 xe^{-x}\,dx$ **(4)**

4. Evaluate $\int_0^{\frac{\pi}{6}} x\cos x\,dx$ **(5)**

5. Find $\int x^2e^x dx$ **(5)**

6. Find $\int e^{-x}\cos 2x\;dx$ **(6)**

7. Find $\int 3x\ln x\,dx$ **(4)**

8. Evaluate $\int_0^3 \ln(2x+3)dx$ **(6)**

Mark Scheme

1.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \cos x$ $v = \sin x$	**M1**
$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$	**M1**
$\int x \cos x \, dx = x \sin x + \cos x + c$	**M1**

2.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = sec^2 x$ $v = \tan x$	**M1**
$\int x sec^2 x \, dx = x\tan x - \int \tan x \, dx$	**M1**
$= x \tan x + \int \frac{-\sin x}{\cos x} \, dx$	**M1**
$= x \tan x + \ln\lvert\cos x\rvert + c$	**M1**

3.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = e^{-x}$ $v = -e^{-x}$	**M1**
$\int_0^2 xe^{-x} \, dx = [-xe^{-x}]_0^2 - \int_0^2 -e^{-x} \, dx$ $= [-xe^{-x}]_0^2 + \int_0^2 e^{-x} \, dx$	**M1**
$= [-xe^{-x} - e^{-x}]_0^2$	**M1**
$= (-2e^{-2} - e^{-2}) - (0 - 1)$ $= 1 - 3e^{-2}$	**M1**

4.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \cos x$ $v = \sin x$	**M1**
$\int_0^{\frac{\pi}{6}} x \cos x \, dx = [x \sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x \, dx$	**M1**
$= [x \sin x + \cos x]_0^{\frac{\pi}{6}}$	**M1**
$= (\frac{\pi}{12} + \frac{\sqrt{3}}{2} - (0 + 1)$	**M1**
$= \frac{1}{12}(\pi + 6\sqrt{3} - 12)$	**M1**

5.

$u = x^2$ $\frac{du}{dx} = 2x$ $\frac{dv}{dx} = e^x$ $v = e^x$	**M1**
$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$	**M1**
For $\int 2xe^x\,dx$, $u = 2x$ $\frac{du}{dx} = 2$ $\frac{dv}{dx} = e^x$ $v = e^x$ $\int 2xe^x\,dx = 2xe^x - \int 2e^x dx$	**M1**
$= 2xe^x - 2e^x + c$	**M1**
$\int x^2e^x\,dx = x^2e^x - (2xe^x - 2e^x) + c$ $= e^x(x^2 - 2x + 2) + c$	**M1**

6.

$u = e^{-x}$ $\frac{du}{dx} = - e^{-x}$ $\frac{dv}{dx} = \cos 2x$ $v = \frac{1}{2}\sin 2x$	**M1**
$\int e^{-x}\cos 2x\,dx = \frac{1}{2}e^{-x}\sin 2x - \int -\frac{1}{2}e^{-x}\sin 2x\;dx$ $= \frac{1}{2}e^{-x}\sin 2x + \int \frac{1}{2}e^{-x}\sin 2x\;dx$	**M1**
For $\int \frac{1}{2}e^{-x}\sin 2x\;dx$ $u = \frac{1}{2}e^{-x}$ $\frac{du}{dx} = -\frac{1}{2}e^{-x}$ $\frac{dv}{dx} = \sin 2x$ $v = -\frac{1}{2}\cos 2x$ $\int \frac{1}{2}e^{-x}\sin 2x\;dx = -\frac{1}{4}e^{-x}\cos 2x - \int \frac{1}{4}e^{-x}\cos 2x\,dx$	**M1**
$\int e^{-x}\cos 2x\,dx = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x - \frac{1}{4}\int e^{-x}\cos 2x\,dx$	**M1**
$\frac{5}{4}\int e^{-x}\cos 2x\,dx = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x + c$	**M1**
$\int e^{-x}\cos 2x\,dx = \frac{1}{5}e^{-x}(2\sin 2x - \cos 2x) + c$	**M1**

7.

$u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 3x$ $v = \frac{3}{2}x^2$	**M1**
$\int 3x\ln x\,dx = \frac{3}{2}x^2\ln x - \int \frac{1}{x}\times\frac{3}{2}x^2\,dx$	**M1**
$= \frac{3}{2}x^2\ln x - \int \frac{3}{2}x\,dx$ $= \frac{3}{2}x^2\ln x - \frac{3}{4}x^2 + c$	**M1**
$= \frac{3}{4}x^2(2\ln x - 1) + c$	**M1**

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8.

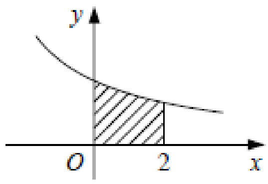
$u = \ln(2x+3)$ $\frac{du}{dx} = \frac{2}{2x+3}$ $\frac{dv}{dx} = 1$ $v = x$	**M1**
$\int_0^3 \ln(2x+3)\,dx = [x\ln(2x+3)]_0^3 - \int_0^3 \frac{2x}{2x+3}\,dx$	**M1**
$= [x\ln(2x+3)]_0^3 - \int_0^3 \frac{(2x+3)-3}{2x+3}\,dx$ $= [x\ln(2x+3)]_0^3 - \int_0^3 1 - \frac{3}{2x+3}\,dx$	**M1**
$= [x\ln(2x+3) - x + \frac{3}{2}\ln\|2x+3\|]_0^3$	**M1**
$= (3\ln 9 - 3 + \frac{3}{2}\ln 9) - (0 - 0 + \frac{3}{2}\ln 3)$	**M1**
$= \frac{15}{2}\ln 3 - 3$	**M1**

Jethwa Maths

Integration
Pt. 7: Parametric Integration

1. The diagram shows part of the curve with parametric equations

$$x = 2t - 4,\ y = \frac{1}{t}$$



The shaded region is bounded by the cruve, the coordinate axes and the line $x = 2$

a. Find the value of the parameter t when $x = 0$, and when $x = 2$ **(1)**

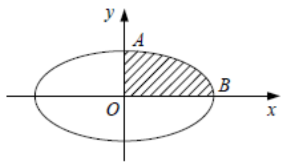
b. Show that the area of the shaded region is given by $\int_2^3 \frac{2}{t}\ dt$ **(2)**

c. Hence, find the area of the shaded region **(2)**

d. Vertify your answer to part c by first finding a cartesian equation fo the curve. **(4)**

2. The diagram shows the ellipse with parametric equations

$$x = 4 \cos \theta,\ y = 2 \sin \theta,\ 0 \leq \theta < 2\pi$$



which meets the positive coordinate axes at the points A and B.

a. Find the value of the parameter θ at the points A and B. **(2)**

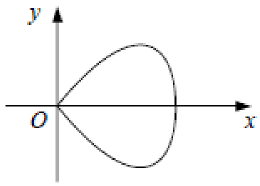
b. Show that the area of the shaded region bounded by the curve and the positive coordinate axes is given by **(2)**

$$\int_0^{\frac{\pi}{2}} 8sin^2\theta\ dx$$

c. Hence, show that the area of the region enclosed by the ellipse is 8π. **(4)**

3. The diagram shows the curve with parametric equations

$$x = 2 \sin t,\ y = 5 \sin 2t,\ 0 \leq t < \pi.$$



a. Show that the area of the region enclosed by the curve is given by **(5)**

$$\int_0^{\frac{\pi}{2}} 20 \sin 2t \cos 2t\ \ dt$$

b. Evaluate this integral. **(4)**

Mark Scheme

1a.

$x = 0$, $t = \quad 2$ $x = 2$, $t = \quad 3$	**M1**

1b.

Area = $\int_0^2 y\ dx$ $x = 2t - 4$ $\frac{dx}{dt} = 2$	**M1**
Area = $\int_2^3 \frac{1}{t} \times 2\ dt$ $= \int_2^3 \frac{2}{t} dt$	**M1**

1c.

$[2 \ln\lvert t\rvert]_2^3$	**M1**
$= 2 \ln 3 - 2 \ln 2$ $= 2 \ln \frac{3}{2}$	**M1**

1d.

$t = \frac{x+4}{2}$ $y = \frac{2}{x+4}$	**M1**
Therefore area = $\int_0^2 \frac{2}{x+4} dx$	**M1**
$= [2 \ln\lvert x+4\rvert]_0^2$	**M1**
$= 2 \ln 6 - 2 \ln 4$ $= 2 \ln \frac{3}{2}$	**M1**

2a.

$x = 0$, $\cos\theta = 0$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ For $y > 0$, $\theta = \frac{\pi}{2}$ at A	**M1**
$y = 0$, $\sin\theta = 0$ $\theta = 0, \pi$ For $x > 0$, $\theta = 0$ at B	**M1**

2b.

$x = 4\cos\theta$ $\frac{dx}{d\theta} = -4\sin\theta$	**M1**
Area = $\int_{\frac{\pi}{2}}^{0} 2\sin\theta \times -4\sin\theta\ d\theta$ $= \int_0^{\frac{\pi}{2}} 8sin^2\theta\ d\theta$	**M1**

2c.

Shaded area = $\int_0^{\frac{\pi}{2}}(4 - 4\cos 2\theta)\ d\theta$	**M1**
= $[4\theta - 2\sin 2\theta]_0^{\frac{\pi}{2}}$	**M1**
= $(2\pi - 0) - (0 - 0)$ = 2π	**M1**
Area of ellipse = 4 x 2π = 8π	**M1**

3a.

y = 0, sin 2t = 0 $t = 0, \frac{\pi}{2}$	**M1**
x = 2 sin t $\frac{dx}{dt} = 2\cos t$	**M1**
Area above x-axis: $=\int_0^{\frac{\pi}{2}} 5\sin 2t \times 2\cos t\ dt$ $= \int_0^{\frac{\pi}{2}} 10\sin 2t\ \cos t\ dt$	**M1**
Area enclosed by curve, $= 2\int_0^{\frac{\pi}{2}} 10\sin 2t\ \cos t\ dt$	**M1**
$= \int_0^{\frac{\pi}{2}} 20\sin 2t\ \cos t\ dt$	**M1**

3b.

$= 40\int_0^{\frac{\pi}{2}} \sin t\ cos^2 t\ dt$	**M1**
$= -40\int_0^{\frac{\pi}{2}} (-\sin t)\ cos^2 t\ dt$	**M1**
$= -40\ [\frac{1}{3}cos^3 t]_0^{\frac{\pi}{2}}$	**M1**
$= -\frac{40}{3}(0 - 1)$ $= 13\frac{1}{3}$	**M1**

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Integration
Pt. 8: Differential Equations

1a. Find $\int (4y+3)^{-\frac{1}{2}}\, dy$ **(4)**

b. Give that $y = 1.5$ at $x = -2$, solve the differential equation, $\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$ **(6)**

2. Find $\int \frac{9x+6}{x}\, dx$, $x > 0$ **(2)**

b. Given that $y = 8$, $x = 1$, solve the differential equation, $\frac{dy}{dx} = \frac{(9x+6)y^{\frac{1}{3}}}{x}$, giving your answer in the form $y = g(x)$ **(6)**

3. Express $\frac{2}{4-y^2}$ in partial fractions. **(3)**

b. Hence, obtain the solution of, $2 \cot x \frac{dy}{dx} = (4 - y^2)$, for which $y = 0$, $x = \frac{\pi}{3}$, giving your answer in the form $\sec^2 x = g(y)$ **(7)**

4. Liquid is pouring into a large vertical cylinder at a constant rate of 1600 cm^3s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2

a. Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation, $\frac{dh}{dt} = 0.4 - k\sqrt{h}$, where k is a positive constants. **(3)**

When h = 25, water is leaking out of the hole at 400cm^3s^{-1}.

b. Show that k = 0.02 **(1)**

c. Separate the variables of the differential equation, $\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$, to show that the time taken to fill the cylinder from empty to a height of 100cm is given by $\int_0^{100} \frac{50}{20-\sqrt{h}}\, dh$ **(2)**

Using the substitution $h = (20 - x)^2$, or otherwise,

d. Find the exact value of $\int_0^{100} \frac{50}{20-\sqrt{h}}\, dh$ **(6)**

e. Hence, find the time taken to fill the cylinder from empty to a height of 100cm, giving your answer in minutes and seconds to the nearest second. **(1)**

5a. Express $\frac{2x-1}{(x-1)(2x-3)}$ in partial fractions **(3)**

b. Given that $x \geq 2$, find the general solution of the differential equation, $(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$ **(5)**

c. Hence, find the particulat solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$. **(4)**

<u>**Mark Scheme**</u>

1a.

Let $u = 4y + 3$ $\frac{du}{dx} = 4$	**M1**
$\int (4y+3)^{-\frac{1}{2}}\,dy = \frac{1}{4}\int u^{-\frac{1}{2}}\,du$	**M1**
$\frac{1}{4}\left(2u^{\frac{1}{2}}\right) + c$	**M1**
$= \frac{1}{2}\left((4y + 3)^{\frac{1}{2}}\right) + c$	**M1**

1b.

$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$ $\frac{1}{\sqrt{4y+3}}\frac{dy}{dx} = \frac{1}{x^2}$	**M1**
$\int \frac{1}{\sqrt{4y+3}}\,dy = \int \frac{1}{x^2}\,dx$ $\int (4y+3)^{-\frac{1}{2}}dy = \int x^{-2}\,dx$	**M1**
$\frac{1}{2}(4y+3)^{\frac{1}{2}} = \frac{x^{-1}}{-1} + c$	**M1**
When $x = -2$, $y = 1.5$ $\frac{3}{2} = \frac{1}{2} + c$ $c = 1$	**M1**
$\frac{1}{2}(4y+3)^{\frac{1}{2}} = 1 - \frac{1}{x}$ $(4y+3)^{\frac{1}{2}} = 2 - \frac{2}{x}$ $4y + 3 = (2 - \frac{2}{x})^2$	**M1**
$4y = (2 - \frac{2}{x})^2 - 3$ $y = \frac{1}{2}[(2 - \frac{2}{x})^2 - 2]$	**M1**

2a.

$\int \frac{9x+6}{x}\,dx = \int (\frac{9x}{x} + \frac{6}{x})\,dx$ $= \int (9 + \frac{6}{x})\,dx$	**M1**
$= 9x + 6\ln x + c$	**M1**

2b.

$\int \frac{1}{y^{\frac{1}{3}}}\,dy = \int \frac{9x+6}{x}\,dx$ $\int y^{-\frac{1}{3}}\,dy = \int \frac{9x+6}{x}\,dx$	**M1**
$\frac{3}{2}y^{\frac{2}{3}} = 9x + 6lnx + c$	**M1**
When $y = 8$, $x = 1$, $6 = 9 + c$ $c = 3$	**M1**
$\frac{3}{2}y^{\frac{2}{3}} = 9x + 6lnx + 3$ $\frac{1}{2}y^{\frac{2}{3}} = 3x + 2\ln x - 1$	**M1**
$y^{\frac{2}{3}} = 6x + 4\ln x - 2$	
$y^2 = (6x + 4\ln x - 2)^3$	**M1**

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3a.

$\frac{2}{4-y^2} = \frac{2}{(2-y)(2+y)} = \frac{A}{2-y} + \frac{B}{2+y}$ 2 = A(2 + y) + B(2 – y)	**M1**
Let y = -2, 2 = 4B $B = \frac{1}{2}$	**M1**
Let y = 2, 2 = 4A $A = \frac{1}{2}$	**M1**
$\frac{2}{(2-y)(2+y)} = \frac{1}{2(2-y)} + \frac{1}{2(2+y)}$	

3b.

$\int \frac{2}{4-y^2}\ dy = \int \frac{1}{\cot x}\ dx$	**M1**
$\int \frac{1}{2(2-y)} + \frac{1}{2(2-y)}\ dy = \int \tan x\ dx$ $\frac{1}{2}\int \frac{1}{2-y} + \frac{1}{2+y}\ dy = \int \tan x\ dx$	**M1**
$\frac{1}{2}[-ln\|2-y\| + ln\|2+y\| = ln\|\sec x\| + c$	**M1**
When y = 0, $x = \frac{\pi}{3}$ $\frac{1}{2}\ln\left(\frac{2}{2}\right) = \ln 2 + c$ c = - ln 2	**M1**
$\frac{1}{2}\ln\frac{2+y}{2-y} = \ln\|\sec x\| - \ln 2$ $\ln\left\|\sqrt{\frac{2+y}{2-y}}\right\| = \ln\|\sec x\|$	**M1**
$\sqrt{\frac{2+y}{2-y}} = \sec x$	**M1**
$\sec^2 x = 4(\frac{2+y}{2-y})$	**M1**

4a.

$\frac{dV}{dt}in = 1600cm^3s^{-1}$ $\frac{dV}{dt}out = -A\sqrt{h}$ $\frac{dV}{dt} = 1600 - A\sqrt{h}$ $V = 4000h$ $\frac{dV}{dh} = 4000$	**M1**
$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\frac{dh}{dt} = \frac{1}{4000} \times (1600 - A\sqrt{h})$	**M1**
$\frac{dh}{dt} = 0.4 - \frac{A}{4000}\sqrt{h}$ where $k = \frac{A}{4000}$	**M1**

4b.

Given h = 25, $\frac{dV}{dt}out = -400$ $-400 = -A\sqrt{25}$ $A = 80$ $k = \frac{80}{4000} = 0.02$	**M1**

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4c.

$\frac{dh}{dt} = (0.4 - 0.02\sqrt{h})$ $\int_0^{100} \frac{1}{0.4-0.02\sqrt{h}}\, dh = \int_0^t dt$	**M1**
$[t]_0^t = \int_0^{100} \frac{1}{0.4-0.02\sqrt{h}}\, dh$ $t = \int_0^{100} \frac{50}{20-\sqrt{h}}\, dh$	**M1**

4d.

$h = (20 - x)^2$ $\sqrt{h} = 20 - x$ $x = 20 - \sqrt{h}$ When $h = 0$, $x = 20$, $h = 100$, $x = 10$	**M1**
$\frac{dx}{dh} = -\frac{1}{2}h^{-\frac{1}{2}} = -\frac{1}{2}\sqrt{h}$ $\frac{dh}{dx} = -2\sqrt{h} = -2(20 - x) = 2(x - 20)$	**M1**
$\int_0^{100} \frac{50}{20-\sqrt{h}}\, dh = \int_{20}^{10} \frac{50}{x}\frac{dh}{dx} dx$ $= \int_{20}^{10} \frac{50}{x} 2(x-20) dx$ $= 100 \int_{20}^{10} \frac{x-20}{x}\, dx$ $= 100 \int_{20}^{10} 1 - \frac{20}{x}\, dx$	**M1**
$= 100[x - 20\ln x]_{20}^{10}$ $= 100[(10 - 20\ln 10) - (20 - 20\ln 20)]$ $= 100[20 \ln 20 - 20 \ln 10 - 10]$	**M1**
$= 100[20 (\ln \frac{20}{10}) - 10]$ $= 100[20 (\ln 2) - 10]$	**M1**
$= 1000(2 (\ln 2) - 1)$ $= 1000(\ln 4 - 1)$	**M1**

4e.

$t = \int_0^{100} \frac{50}{20-\sqrt{h}}\, dh$ = 1000(ln 4 – 1) dh = 386.29 seconds = 6 minutes and 26 seconds.	**M1**

5a.

$\frac{2x-1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$ $2x - 1 = A(2x - 3) + B(x - 1)$	**M1**
Let $x = 1$, $1 = -A$ $A = -1$	**M1**
Let $x = \frac{3}{2}$ $2 = \frac{1}{2}B$ $B = 4$	**M1**
$\frac{2x-1}{(x-1)(2x-3)} = \frac{-1}{x-1} + \frac{4}{2x-3}$	

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5b.

$(2x - 3)(x - 1)\frac{dy}{dx} = (2x - 1)y$	**M1**
$\int \frac{1}{y}\ dy = \int \frac{2x-1}{(2x-3)(x-1)}\ dx$	**M1**
$\ln\|y\| = \int \frac{4}{2x-3}\ dx - \int \frac{1}{x-1}\ dx$ $\ln\|y\| = 2\ln\|2x-3\| - \ln\|x-1\| + c$	**M1**
$\ln\|y\| = \ln\|(2x-3)^2\| - \ln\|x-1\| + c$	**M1**
$\ln\|y\| = \ln\left\|\frac{(2x-3)^2}{x-1}\right\| + c$	**M1**

5c.

$\ln\|y\| = \ln\left\|\frac{(2x-3)^2}{x-1}\right\| + c$ when $x = 2$, $y = 10$ $\ln 10 = \ln 1 + c$ $c = \ln 10$	**M1**
$\ln\|y\| = \ln\left\|\frac{(2x-3)^2}{x-1}\right\|$	**M1**
$\ln\|y\| = \ln\left\|\frac{10(2x-3)^2}{x-1}\right\|$	**M1**
$y = \frac{10(2x-3)^2}{x-1}$	**M1**

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