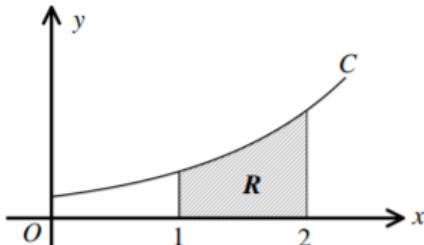


A-Level Unit Test 2: Integration Methods



1. Find the exact value of $\int_0^{\frac{\pi}{4}} x \sin 3x \, dx$ (5)

2. The figure above shows the curve C , given parametrically by, $x = \ln t$, $y = t + \sqrt{t}$, $1 \leq t \leq 10$. The finite region R is bounded by C , the straight lines with equations $x = 1$ and $x = 2$ and the x -axis.



- a. Show that the area of R is given by, (4)

$$\int_{T1}^{T2} 1 + t^{-\frac{1}{2}} \, dt$$

Stating the values of $T1$ and $T2$.

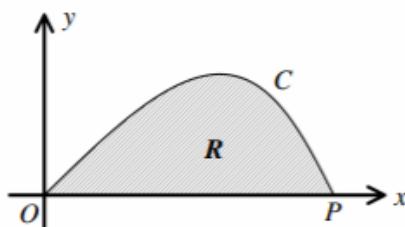
- b. Hence find an exact value for the area of R . (3)

3. Using the substitution $u = x^2 + 2x$, find the $\int (x+1)(x^2 + 2x)^3 \, dx$ (4)

4. Find the exact value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$ (5)

5. Find $\int e^x \sin x \, dx$ (5)

6. The figure shows the curve C , given parametrically by, $x = 3t + \sin t$, $y = 2 \sin t$, $0 \leq t \leq \pi$.



The curve meets the coordinate axes at the point P and at the origin O . The finite region R is bounded by C and the x -axis. Determine the area of R . (5)

7. Water is being heated in a kettle. At time t seconds, the temperature of the water is $\theta^\circ\text{C}$. The rate of increase of the temperature of the water at any time t is modelled by the differential equation,

$$\frac{d\theta}{dt} = \alpha(120 - \theta), \theta \leq 100.$$

Where α is a positive constant.

Given that $\theta = 20$, when $t = 0$.

- a. Solve this differential equation to show that, $\theta = 120 - 100e^{-\alpha t}$ (7)

When the temperature of the water reaches 100°C , the kettle switches off.

- b. Given that $\alpha = 0.01$, find the time, to the nearest second, when the kettle switches off. (3)

8. Find $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x} dx$ when $u = 1 + \cos x$ (5)

9. Find $\int (\ln x)^2 dx$ (4)

10. Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation, $\frac{dy}{dx} = \frac{3}{y \cos^2 x}$

11. Evaluate $\int_2^4 \frac{x+1}{x^2+2x+8} dx$ (4)

12. Find $\int x\sqrt{1-x} dx$ using the substitution $u^2 = 1 - x$ (5)

13a. Express $\frac{1}{P(5-P)}$ in partial fractions. (3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation,

$$\frac{dP}{dt} = \frac{1}{15} P(5 - P), t \geq 0$$

Where P , in thousands, is the population of meerkats and t is the time measured in years since the study began. Given that when $t = 0$, $P = 1$,

b. Solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + ce^{-\frac{1}{3}t}}$$

Where a , b and c are integers. (8)

c. Hence, show that the population cannot exceed 5000. (1)

14. Find the exact value of $\int_{\frac{1}{3}}^1 2xe^{3x-1} dx$ (5)

15. Find the exact value of $\int_0^1 \sqrt{4-x^2} dx$ when $x = 2 \sin u$. (6)

Total marks: 60

Mark Scheme

1.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \sin 3x$ $v = -\frac{1}{3} \cos 3x$	M1
$\int_0^{\frac{\pi}{4}} x \sin 3x \, dx = [-\frac{1}{3} x \cos 3x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{1}{3} \cos 3x \, dx$	M1
$= [-\frac{1}{3} x \cos 3x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{3} \cos 3x \, dx$	M1
$= [-\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x]_0^{\frac{\pi}{4}}$	M1
$= [\frac{\pi}{12}(-\frac{1}{\sqrt{2}}) + \frac{1}{9}(\frac{1}{\sqrt{2}})] - (0)$	M1
$= \frac{1}{72}\sqrt{2}(3\pi + 4)$	M1

2a.

When $x = 1, \ln t = 1$ $t = e$ When $x = 2, \ln t = 2$ $t = e^2$	M1
Area $= \int_{x_1}^{x_2} y(x)dx = \int_{T_1}^{T_2} y(t) \frac{dx}{dt} dt = \int_e^{e^2} (t + t^{\frac{1}{2}})(\frac{1}{t}) dt$	M1
$= \int_e^{e^2} t \times \frac{1}{t} + t^{\frac{1}{2}} \times \frac{1}{t} dt$	M1
$= \int_e^{e^2} 1 + t^{-\frac{1}{2}} dt$	M1

2b.

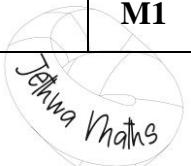
$[t + 2t^{\frac{1}{2}}]_e^{e^2} = [e^2 + 2(e^2)^{\frac{1}{2}}] - [e + 2e^{\frac{1}{2}}]$	M1
$= e^2 + 2e - e - 2e^{\frac{1}{2}}$	M1
$= e^2 + e - 2\sqrt{e}$	M1

3.

$u = x^2 + 2x$ $\frac{du}{dx} = 2x + 2$	M1
$\int (x+1)(x^2+2x)^3 dx = \int \frac{1}{2} u^3 du$	M1
$= \frac{1}{8} u^4 + c$	M1
$= \frac{1}{8} (x^2 + 2x)^4 + c$	M1

4.

Let $u = \tan x$ $\frac{du}{dx} = \sec^2 x$	M1
$x = -\frac{\pi}{4}, u = -1$ $x = \frac{\pi}{4}, u = 1$	M1
$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x dx = \int_{-1}^1 u^2 du$	M1



$= \left[\frac{1}{3} u^3 \right]_1^{-1}$	M1
$= \frac{1}{3} [1 - (-1)]$	M1
$= \frac{2}{3}$	

5.

$u = e^x$	M1
$\frac{du}{dx} = e^x$	
$\frac{dv}{dx} = \sin x$	
$v = -\cos x$	
$\int e^x \sin x \, dx = -e^x \cos x - \int -e^x \cos x \, dx$	M1
$= -e^x \cos x + \int e^x \cos x \, dx$	
For $\int e^x \cos x \, dx$	
$u = e^x$	
$\frac{du}{dx} = e^x$	
$\frac{dv}{dx} = \cos x$	M1
$v = \sin x$	
$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$	
$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$	M1
$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + c$	
$\int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + c$	M1

6.

When $t = 0, x = 0, y = 0$	M1
When $t = \pi, x = 3\pi, y = 0$	
Area = $\int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt = \int_0^\pi (2 \sin t)(3 + \cos t) dt$	M1
$= \int_0^\pi 6 \sin t + 2 \sin t \cos t \, dt$	M1
$= \int_0^\pi 6 \sin t + \sin 2t \, dt$	
$= [-6 \cos t - \frac{1}{2} \cos 2t]_0^\pi$	M1
$= [6 \cos t + \frac{1}{2} \cos 2t]_0^\pi$	
$= (6 + \frac{1}{2}) - (6 + \frac{1}{2})$	M1
$= 12$	

7a.

$\frac{d\theta}{dt} = \alpha(120 - \theta)$	M1
$\int \frac{1}{120-\theta} d\theta = \int \alpha dt$	
$-\ln(120 - \theta) = \alpha t + c$	M1
When $\theta = 20, t = 0$	
$-\ln 100 = c$	M1
$-\ln(120 - \theta) = \alpha t - \ln 100$	
$\ln 100 - \ln(120 - \theta) = \alpha t$	M1
$\ln \frac{100}{120-\theta} = \alpha t$	
$\frac{100}{120-\theta} = e^{\alpha t}$	M1
$100 = e^{\alpha t}(120 - \theta)$	
$\frac{100}{e^{\alpha t}} = 120 - \theta$	M1
$\theta = 120 - 100e^{-\alpha t}$	M1

7b.

$$\text{When } \theta = 100$$

$$100 = 120 - 100e^{-0.0t}$$

$$100e^{-0.01t} = 20$$

$$5 = e^{0.01t}$$

$$\ln 5 = 0.01t$$

$$t = \frac{\ln 5}{0.01}$$

$t = 161$ seconds.

M1**M1****M1**

8.

$$u = 1 + \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$x = 0, u = 2$$

$$x = \frac{\pi}{2}, u = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x} dx = \int_2^1 -\frac{1}{u} du = \int_1^2 \frac{1}{u} du$$

$$= [\ln |u|]_1^2$$

$$= \ln 2 - 0$$

$$= \ln 2$$

M1**M1****M1****M1****M1**

9.

$$u = (\ln x)^2$$

$$\frac{du}{dx} = 2(\ln x) \times \frac{1}{x}$$

$$\frac{dv}{dx} = 1$$

$$v = x$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2 \ln x dx$$

For $\int 2 \ln x dx$,

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 2$$

$$v = 2x$$

$$\int 2 \ln x dx = 2x \ln x - \int 2 dx$$

$$= 2x \ln x - 2x + c$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - (2x \ln x - 2x) + c$$

$$= x[(\ln x)^2 - 2 \ln x + 2] + c$$

M1**M1****M1****M1****M1**

10.

(5)

$$\frac{dy}{dx} = \frac{3}{y \cos^2 x}$$

$$\int y dy = \int \frac{3}{\cos^2 x} dx$$

$$\int y dy = 3 \int \sec^2 x dx$$

$$\frac{y^2}{2} = 3 \tan x + c$$

$$\text{When } x = \frac{\pi}{4}, y = 2$$

$$2 = 3 + c$$

$$c = -1$$

$$\frac{y^2}{2} = 3 \tan x - 1$$

M1**M1****M1****M1**

11.

$\int_2^4 \frac{x+1}{x^2+2x+8} dx = \frac{1}{2} \int_2^4 \frac{2x+2}{x^2+2x+8} dx$	M1
$= \frac{1}{2} [\ln x^2 + 2x + 8]_2^4$	M1
$= \frac{1}{2} (\ln 32 - \ln 16)$	M1
$= \frac{1}{2} \ln 2$	M1

12.

$u^2 = 1 - x \rightarrow x = 1 - u^2$	M1
$\frac{dx}{du} = -2u$	M1
$\int x\sqrt{1-x} dx = \int (1-u^2)u \times (-2u) du$	M1
$= 2 \int (u^4 - u^2) du$	M1
$= 2(\frac{1}{5}u^5 - \frac{1}{3}u^3) + c$	M1
$= 2[\frac{1}{5}(1-x)^{\frac{5}{2}} - \frac{1}{3}(1-x)^{\frac{3}{2}}] + c$	M1
$= \frac{2}{15}(1-x)^{\frac{3}{2}}[3(1-x) - 5] + c$	M1
$= \frac{2}{15}(2+3x)(1-x)^{\frac{3}{2}} + c$	M1

15.

$x = 2 \sin u$	M1
$\frac{dx}{du} = 2 \cos u$	M1
$x = 0, u = 0$	M1
$x = 1, u = \frac{\pi}{6}$	M1
$\int_0^1 \sqrt{4-x^2} dx = \int_0^{\frac{\pi}{6}} 2 \cos u \times 2 \cos u du$	M1
$= \int_0^{\frac{\pi}{6}} 4 \cos^2 u du$	M1
$= \int_0^{\frac{\pi}{6}} (2 + 2 \cos 2u) du$	M1
$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$	M1
$= (\frac{\pi}{3} + \frac{\sqrt{3}}{2}) - (0)$	M1
$= \frac{1}{6}(2\pi + 3\sqrt{3})$	M1