

A-Level Unit Test 1: Integration



1. Find the exact value of $y = 2 - \frac{1}{2}e^x$ when is bound by $x = 0$ and $x = \ln 2$ (3)

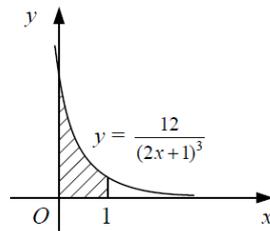
2a. Sketch the curve $y = e^x - a$ where a is a constant and $a > 1$. (4)

Show on your sketch the coordinates of any points of intersection with the coordinate axes and the equation of any asymptotes.

b. Find, in terms of a , the area of the finite region bounded by the curve $y = e^x - a$ and the coordinate axes. (2)

c. Given that the area of this region is $1 + a$, show that $a = e^2$. (3)

3. The diagram shows part of the curve with equation $y = \frac{12}{(2x+1)^3}$.



Find the area of the shaded region bounded by the curve, the coordinate axes and the line $x = 1$. (5)

4. Integrate $\frac{3x^2-5}{x^2-1}$ with respect to x (7)

5. Find the exact value of $\int_0^2 \frac{2x^2-7x+7}{x^2-2x-3} dx$ (8)

6. Evaluate $\int_0^1 \frac{3}{2x^2-8} dx$ (3)

7. Integrate $\cos(2x-1)$ with respect to x . (2)

8. Integrate $\operatorname{cosec} \frac{1}{4}x \cot \frac{1}{4}x$ with respect to x . (2)

9. Evaluate $\int_0^{\frac{\pi}{2}} \sin 2x dx$ (3)

10. Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 3x dx$ (3)

11a. Express $\tan^2 x$ in terms of $\sec x$. (1)

b. Show that $\int \tan^2 x dx = \tan x - x + c$ (2)

12. Find $\int \sin x \cos x dx$ (2)

13. Find $\int (\sec x - \tan x)^2 dx$ (3)

14. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos 2x}{\sin^2 2x} dx$ (4)

15. Evaluate $\int_0^{\frac{\pi}{4}} (1 - 2 \sin x)^2 dx$ (4)

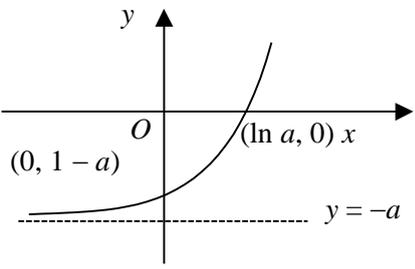
Total marks: 60

Mark Scheme

1.

$\int_0^{\ln 2} 2 - \frac{1}{2}e^x dx = [2x - \frac{1}{2}e^x]_0^{\ln 2}$	M1
$= (2 \ln 2 - 1) - (0 - \frac{1}{2})$	M1
$= 2 \ln 2 - \frac{1}{2}$	M1

2a.

Shape M1 Intercepts M1 M1 Asymptotes M1		
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2b.

$-\int_0^{\ln a} (e^x - a) dx = -[e^x - ax]_0^{\ln a}$	M1
$= -[(a - a \ln a) - (1 - 0)] = 1 - a + a \ln a$	M1

2c.

$1 - a + a \ln a = 1 + a$	M1
$a \ln a = 2a$	M1
$\ln a = 2$	M1
$a = e^2$	M1

3.

$\int_0^1 12(2x + 1)^{-3} dx$	M1
$= [\frac{1}{2} \times (-6)(2x + 1)^{-2}]_0^1$	M1
$= [\frac{-3}{(2x+1)^2}]_0^1$	M1
$= -\frac{1}{3} - (-3)$	M1
$= \frac{8}{3}$	M1

4.

$\frac{3x^2-5}{x^2-1} = A + \frac{B}{x+1} + \frac{C}{x-1}$	M1
$3x^2 - 5 = A(x+1)(x-1) + B(x-1) + C(x+1)$	M1
Let $x = -1$ $-2 = -2B$ $B = 1$	M1
Let $x = 1$ $-2 = 2C$ $C = -1$	M1
Coefficients of x^2 , $A = 3$	M1
$\int \frac{3x^2-5}{x^2-1} dx = \int 3 + \frac{1}{x+1} - \frac{3}{x-1} dx$	M1
$= 3x + \ln x+1 - \ln x-1 + c$	M1

5.

$\frac{2x^2-7x+7}{x^2-2x-3} = A + \frac{B}{x-3} + \frac{C}{x+1}$	M1
$2x^2 - 7x + 7 = A(x-3)(x+1) + B(x+1) + C(x-3)$	M1
Let $x = 3$, $4 = 4B$ $B = 1$	M1
Let $x = -1$, $16 = -4C$ $C = -4$	M1
Coefficients of x^2 , $A = 2$	M1
$\int_0^2 \frac{2x^2-7x+7}{x^2-2x-3} dx = \int_0^2 2 + \frac{1}{x-3} - \frac{4}{x+1} dx$	M1
$= [2x + \ln x-3 - 4 \ln x+1]_0^2$	M1
$= (4 + 0 - 4 \ln 3) - (0 + \ln 3 - 0)$ $= 4 - 5 \ln 3$	M1

6.

$\int_0^1 \frac{3}{2x^2-8} dx$ $= \frac{3}{2} \left[\frac{1}{4} \ln \left \frac{x-2}{x+2} \right \right]_0^1$	M1
$= \frac{3}{8} (\ln \frac{1}{3} - 0)$	M1
$= -\frac{3}{8} \ln 3$	M1

7.

$\int \cos(2x-1) dx = \frac{1}{2} \sin(2x-1) + c$	M1 M1
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8.

$\int \operatorname{cosec} \frac{1}{4}x \cot \frac{1}{4}x dx = -4 \operatorname{cosec} \frac{1}{4}x + c$	M1 M1
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9.

$\int_0^{\frac{\pi}{2}} \sin 2x dx = \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{6}}$	M1 M1
$= \frac{1}{4} - \left(-\frac{1}{2} \right) = \frac{3}{4}$	M1

10.

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 3x dx = \left[\frac{1}{3} \tan 3x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$	M1 M1
$= 0 - \left(-\frac{1}{3} \right) = \frac{1}{3}$	M1

11a.

$\tan^2 x = \sec^2 x - 1$	M1
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11b.

$\int \tan^2 x dx = \int \sec^2 x - 1 dx = \tan x - x + c$	M1 M1
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12.

$\int \sin x \cos x dx = \int \frac{1}{2} \sin 2x dx$	M1
$= -\frac{1}{4} \cos 2x + c$	M1

13.

$\int (\sec x - \tan x)^2 dx = \int (\sec^2 x - 2 \sec x \tan x + \tan^2 x) dx$	M1
$= \int (\sec^2 x - 2 \sec x \tan x + \sec^2 x - 1) dx$	M1
$= \int (2\sec^2 x - 2 \sec x \tan x - 1) dx$	M1
$= 2 \tan x - 2 \sec x - x + c$	M1

14.

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos 2x}{\sin^2 2x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin 2x} \times \frac{\cos 2x}{\sin 2x} dx$	M1
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} 2x \cot 2x dx$	M1
$= \left[-\frac{1}{2} \operatorname{cosec} 2x \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$	M1
$= -\frac{1}{2} - \left(-\frac{1}{\sqrt{3}} \right) = \frac{1}{3} \sqrt{3} - \frac{1}{2}$	M1

15.

$\int_0^{\frac{\pi}{4}} (1 - 2 \sin x)^2 dx = \int_0^{\frac{\pi}{4}} (1 - 4 \sin x + 4 \sin^2 2x) dx$	M1
$= \int_0^{\frac{\pi}{4}} [1 - 4 \sin x + 2(1 - \cos 2x)] dx$	M1
$= \int_0^{\frac{\pi}{4}} (3 - 4 \sin x - 2 \cos 2x) dx$	M1
$= [3x + 4 \cos x - \sin 2x]_0^{\frac{\pi}{4}}$	M1
$= \left(\frac{3\pi}{4} + 2\sqrt{2} - 1 \right) - (0 + 4 - 0)$	M1
$= \frac{3\pi}{4} + 2\sqrt{2} - 5$	M1

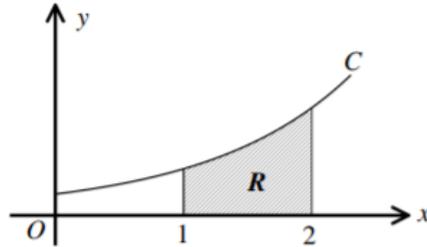


A-Level Unit Test 2: Integration Methods



1. Find the exact value of $\int_0^{\frac{\pi}{4}} x \sin 3x \, dx$ (5)

2. The figure above shows the curve C , given parametrically by, $x = \ln t$, $y = t + \sqrt{t}$, $1 \leq t \leq 10$. The finite region R is bounded by C , the straight lines with equations $x = 1$ and $x = 2$ and the x -axis.



a. Show that the area of R is given by, (4)

$$\int_{T_1}^{T_2} 1 + t^{-\frac{1}{2}} \, dt$$

Stating the values of T_1 and T_2 .

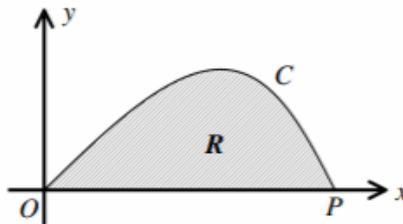
b. Hence find an exact value for the area of R . (3)

3. Using the substitution $u = x^2 + 2x$, find the $\int (x + 1)(x^2 + 2x)^3 \, dx$ (4)

4. Find the exact value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$ (5)

5. Find $\int e^x \sin x \, dx$ (5)

6. The figure shows the curve C , given parametrically by, $x = 3t + \sin t$, $y = 2 \sin t$, $0 \leq t \leq \pi$.



The curve meets the coordinate axes at the point P and at the origin O . The finite region R is bounded by C and the x -axis. Determine the area of R . (5)

7. Water is being heated in a kettle. At time t seconds, the temperature of the water is $\theta^\circ\text{C}$. The rate of increase of the temperature of the water at any time t is modelled by the differential equation,

$$\frac{d\theta}{dt} = \alpha(120 - \theta), \theta \leq 100.$$

Where α is a positive constant.

Given that $\theta = 20$, when $t = 0$.

a. Solve this differential equation to show that, $\theta = 120 - 100e^{-\alpha t}$ (7)

When the temperature of the water reaches 100°C , the kettle switches off.

b. Given that $\alpha = 0.01$, find the time, to the nearest second, when the kettle switches off. (3)

8. Find $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x} dx$ when $u = 1 + \cos x$ (5)

9. Find $\int (\ln x)^2 dx$ (4)

10. Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation, $\frac{dy}{dx} = \frac{3}{y \cos^2 x}$

11. Evaluate $\int_2^4 \frac{x+1}{x^2+2x+8} dx$ (4)

12. Find $\int x\sqrt{1-x} dx$ using the substitution $u^2 = 1-x$ (5)

13a. Express $\frac{1}{P(5-P)}$ in partial fractions. (3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation,

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), t \geq 0$$

Where P , in thousands, is the population of meerkats and t is the time measured in years since the study began. Given that when $t = 0$, $P = 1$,

b. Solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b+ce^{-\frac{1}{3}t}}$$

Where a , b and c are integers. (8)

c. Hence, show that the population cannot exceed 5000. (1)

14. Find the exact value of $\int_{\frac{1}{3}}^1 2xe^{3x-1} dx$ (5)

15. Find the exact value of $\int_0^1 \sqrt{4-x^2} dx$ when $x = 2 \sin u$. (6)

Total marks: 60

Mark Scheme

1.

$u = x$ $\frac{du}{dx} = 1$ $\frac{dv}{dx} = \sin 3x$ $v = -\frac{1}{3} \cos 3x$	M1
$\int_0^{\frac{\pi}{4}} x \sin 3x \, dx = \left[-\frac{1}{3} x \cos 3x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{1}{3} \cos 3x \, dx$ $= \left[-\frac{1}{3} x \cos 3x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{3} \cos 3x \, dx$	M1
$= \left[-\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x\right]_0^{\frac{\pi}{4}}$	M1
$= \left[-\frac{\pi}{12} \left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{9} \left(\frac{1}{\sqrt{2}}\right)\right] - (0)$	M1
$= \frac{1}{72} \sqrt{2} (3\pi + 4)$	M1

2a.

When $x = 1$, $\ln t = 1$ $t = e$ When $x = 2$, $\ln t = 2$ $t = e^2$	M1
$\text{Area} = \int_{x=1}^{x=2} y(x) \, dx = \int_{t=e}^{t=e^2} y(t) \frac{dx}{dt} \, dt = \int_e^{e^2} \left(t + t^{\frac{1}{2}}\right) \left(\frac{1}{t}\right) \, dt$	M1
$= \int_e^{e^2} t \times \frac{1}{t} + t^{\frac{1}{2}} \times \frac{1}{t} \, dt$	M1
$= \int_e^{e^2} 1 + t^{-\frac{1}{2}} \, dt$	M1

2b.

$\left[t + 2t^{\frac{1}{2}}\right]_e^{e^2} = \left[e^2 + 2(e^2)^{\frac{1}{2}}\right] - \left[e + 2e^{\frac{1}{2}}\right]$	M1
$= e^2 + 2e - e - 2e^{\frac{1}{2}}$	M1
$= e^2 + e - 2\sqrt{e}$	M1

3.

$u = x^2 + 2x$ $\frac{du}{dx} = 2x + 2$	M1
$\int (x+1)(x^2+2x)^3 \, dx = \int \frac{1}{2} u^3 \, du$	M1
$= \frac{1}{8} u^4 + c$	M1
$= \frac{1}{8} (x^2 + 2x)^4 + c$	M1

4.

Let $u = \tan x$ $\frac{du}{dx} = \sec^2 x$	M1
$x = -\frac{\pi}{4}, u = -1$ $x = \frac{\pi}{4}, u = 1$	M1
$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx = \int_{-1}^1 u^2 \, du$	M1

$= \left[\frac{1}{3} u^3 \right]_{-1}^1$	M1
$= \frac{1}{3} [1 - (-1)]$	M1
$= \frac{2}{3}$	

5.

$u = e^x$ $\frac{du}{dx} = e^x$ $\frac{dv}{dx} = \sin x$ $v = -\cos x$	M1
$\int e^x \sin x dx = -e^x \cos x - \int -e^x \cos x dx$ $= -e^x \cos x + \int e^x \cos x dx$	M1
For $\int e^x \cos x dx$ $u = e^x$ $\frac{du}{dx} = e^x$ $\frac{dv}{dx} = \cos x$ $v = \sin x$ $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$	M1
$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$	M1
$2 \int e^x \sin x dx = -e^x \cos x + e^x \sin x + c$ $\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + c$	M1

6.

When $t = 0, x = 0, y = 0$ When $t = \pi, x = 3\pi, y = 0$	M1
$\text{Area} = \int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt = \int_0^\pi (2 \sin t)(3 + \cos t) dt$	M1
$= \int_0^\pi 6 \sin t + 2 \sin t \cos t dt$ $= \int_0^\pi 6 \sin t + \sin 2t dt$	M1
$= [-6 \cos t - \frac{1}{2} \cos 2t]_0^\pi$ $= [6 \cos t + \frac{1}{2} \cos 2t]_\pi^0$	M1
$= (6 + \frac{1}{2}) - (6 + \frac{1}{2})$ $= 12$	M1

7a.

$\frac{d\theta}{dt} = \alpha(120 - \theta)$ $\int \frac{1}{120 - \theta} d\theta = \int \alpha dt$	M1
$-\ln(120 - \theta) = \alpha t + c$ When $\theta = 20, t = 0$	M1
$-\ln 100 = c$	M1
$-\ln(120 - \theta) = \alpha t - \ln 100$ $\ln 100 - \ln(120 - \theta) = \alpha t$ $\ln \frac{100}{120 - \theta} = \alpha t$	M1
$\frac{100}{120 - \theta} = e^{\alpha t}$	M1
$100 = e^{\alpha t}(120 - \theta)$	M1
$\frac{100}{e^{\alpha t}} = 120 - \theta$	M1
$\theta = 120 - 100e^{-\alpha t}$	M1

7b.

When $\theta = 100$ $100 = 120 - 100e^{-0.01t}$ $100e^{-0.01t} = 20$	M1
$5 = e^{0.01t}$	M1
$\ln 5 = 0.01t$ $t = \frac{\ln 5}{0.01}$ $t = 161$ seconds.	M1

8.

$u = 1 + \cos x$ $\frac{du}{dx} = -\sin x$	M1
$x = 0, u = 2$ $x = \frac{\pi}{2}, u = 1$	M1
$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx = \int_2^1 -\frac{1}{u} du = \int_1^2 \frac{1}{u} du$	M1
$= [\ln u]_1^2$	M1
$= \ln 2 - 0$ $= \ln 2$	M1

9.

$u = (\ln x)^2$ $\frac{du}{dx} = 2(\ln x) \times \frac{1}{x}$ $\frac{dv}{dx} = 1$ $v = x$	M1
$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2 \ln x dx$	M1
For $\int 2 \ln x dx$, $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 2$ $v = 2x$ $\int 2 \ln x dx = 2x \ln x - \int 2 dx$ $= 2x \ln x - 2x + c$	M1
$\int (\ln x)^2 dx = x(\ln x)^2 - (2x \ln x - 2x) + c$ $= x[(\ln x)^2 - 2 \ln x + 2] + c$	M1

10.

(5)

$\frac{dy}{dx} = \frac{3}{y \cos^2 x}$ $\int y dy = \int \frac{3}{\cos^2 x} dx$	M1
$\int y dy = 3 \int \sec^2 x dx$	M1
$\frac{y^2}{2} = 3 \tan x + c$	M1
When $x = \frac{\pi}{4}, y = 2$ $2 = 3 + c$ $c = -1$	M1
$\frac{y^2}{2} = 3 \tan x - 1$	M1



11.

$\int_2^4 \frac{x+1}{x^2+2x+8} dx = \frac{1}{2} \int_2^4 \frac{2x+2}{x^2+2x+8} dx$	M1
$= \frac{1}{2} [\ln x^2 + 2x + 8]_2^4$	M1
$= \frac{1}{2} (\ln 32 - \ln 16)$	M1
$= \frac{1}{2} \ln 2$	M1

12.

$u^2 = 1 - x \rightarrow x = 1 - u^2$ $\frac{dx}{du} = -2u$	M1
$\int x\sqrt{1-x} dx = \int (1-u^2)u \times (-2u) du$	M1
$= 2\int (u^4 - u^2) du$ $= 2(\frac{1}{5}u^5 - \frac{1}{3}u^3) + c$	M1
$= 2[\frac{1}{5}(1-x)^{\frac{5}{2}} - \frac{1}{3}(1-x)^{\frac{3}{2}}] + c$ $= \frac{2}{15}(1-x)^{\frac{3}{2}}[3(1-x) - 5] + c$	M1
$= -\frac{2}{15}(2+3x)(1-x)^{\frac{3}{2}} + c$	M1

15.

$x = 2 \sin u$ $\frac{dx}{du} = 2 \cos u$	M1
$x = 0, u = 0$ $x = 1, u = \frac{\pi}{6}$	M1
$\int_0^1 \sqrt{4-x^2} dx = \int_0^{\frac{\pi}{6}} 2 \cos u \times 2 \cos u du$ $= \int_0^{\frac{\pi}{6}} 4 \cos^2 u du$	M1
$= \int_0^{\frac{\pi}{6}} (2 + 2 \cos 2u) du$	M1
$= [2u + \sin 2u]_0^{\frac{\pi}{6}}$	M1
$= (\frac{\pi}{3} + \frac{\sqrt{3}}{2}) - (0)$ $= \frac{1}{6}(2\pi + 3\sqrt{3})$	M1

