A-Level Unit Test: Differentiation

1. Given that $y = x(2x+1)^4$, show that $\frac{dy}{dx} = (2x+1)^n(Ax+B)$ where n, A and B are constants to be found. **(5)**

2. The curve C has equation y = f(x) where, $f(x) = \frac{4x+1}{x-2}$, $x > 2$.

a. Show that $f'(x) = \frac{-9}{(x-2)^2}$ **(3)**

Given that P is a point on C such that $f'(x) = -1$.

b. Find the coordinates of P **(4)**

3. Given that $x = \sec^2 3y$, $0 < y < \frac{\pi}{6}$

a. Find $\frac{dx}{dy}$ in terms of y. **(2)**

b. Hence show that $\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}$ **(3)**

c. Find an expression for $\frac{d^2y}{dx^2}$ in terms of x. Give your answer in its simplest form. **(4)**

4. Given that $\frac{d}{dx}(\cos x) = -\sin x$,

a. Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ **(3)**

Given that $x = \sec 2y$,

b. Find $\frac{dx}{dy}$ in terms of y **(2)**

c. Hence find $\frac{dy}{dx}$ in terms of x **(4)**

5a. Given that $y = \frac{\ln(x^2+1)}{x}$, find $\frac{dy}{dx}$ **(4)**

b. Given that $x = \tan y$, show that $\frac{dy}{dx} = \frac{1}{1+x^2}$ **(3)**

6. The curve C has equation $x = 8y \tan 2y$. The point P has coordinates $(\pi, \frac{\pi}{8})$

a. Verify that P lies on C. **(1)**

b. Find the equation of the tangent to C at P in the form $ay = x + b$, where the constants a and b are to be found in terms of π. **(6)**

7. The point P lies on the curve with equation $y = \ln(\frac{1}{3}x)$. The x-coordinates of P is 3. Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants.

8. The curve C has equation $16y^3 + 9x^2y - 54x = 0$.

a. Find $\frac{dy}{dx}$ in terms of x and y **(5)**

b. Find the coordinates of the points on C where $\frac{dy}{dx} = 0$ **(7)**

9. A curve C has equation $2^x + y^2 = 2xy$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (3, 2) **(7)**

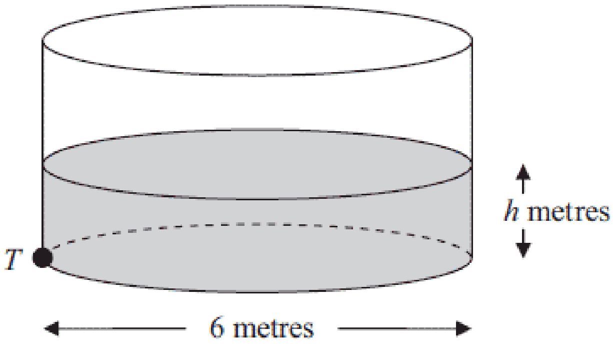
10. The curve C has equation $ye^{-2x} = 2x + y^2$

a. Find $\frac{dy}{dx}$ in terms of x and y **(5)**

The point P on C has coordinates (0, 1).

b. Find the equation of the normal to C at P giving your answers in the form $ax + by + c = 0$, where a, b and c are integers. **(4)**

11. The figure below shows a cylindrical water tank.



The diameter of a circular cross section of the tank is 6m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at aa rate of 0.6πh m^3 min^{-1}.
Show that t minutes after the tap has been opened, $75\frac{dh}{dt} = (4 - 5h)$ **(5)**

12. A curve has parametric equations,

$$x = 7\cos t - \cos 7t$$
$$y = 7\sin t - \sin 7t$$

a. Find an expression for $\frac{dy}{dx}$ in terms of t. You do not need to simplify your answer. **(3)**
b. Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. Give your answer in its simplest form. **(5)**

Total marks: 85

Mark Scheme

1.

$y = x(2x + 1)^4$ $\frac{dy}{dx} = x \times 4(2x + 1)^3(2) + (2x + 1)^4(1)$	**M1 M1**
$\frac{dy}{dx} = 8x(2x + 1)^3 + (2x + 1)^4$	**M1**
$\frac{dy}{dx} = (2x + 1)^3[8x + (2x + 1)]$	**M1**
$\frac{dy}{dx} = (2x + 1)^3[10x + 1)]$ $n = 3, A = 10, B = 1$	**M1**

2a.

Using quotient rule: $u = 4x + 1$ $u' = 4$ $v = x - 2$ $v' = 1$	**M1**
$f'(x) = \frac{(x-2)(4)-(4x+1)(1)}{(x-2)^2}$	**M1**
$f'(x) = \frac{4x-8-4x-1}{(x-2)^2} = \frac{-9}{(x-2)^2}$	**M1**

2b.

$f'(x) = -1$ $-1 = \frac{-9}{(x-2)^2}$	**M1**
$(x - 2)^2 = 9$ $x - 2 = \pm 3$ $x = 5$ or -1	**M1**
As $x > 2$, $x = 5$.	**M1**
When $x = 5$, $f(5) = \frac{4(5)+1}{5-2} = 7$ P: (5, 7)	**M1**

3a.

$x = \sec^2 y = (\sec 3y)^2$ $\frac{dx}{dy} = 2(\sec 3y)(3 \sec 3y)(\tan 3y)$	**M1**
$\frac{dx}{dy} = 6\sec^2 3y \tan 3y$	**M1**

3b.

$1 + \tan^2 x = \sec^2 x$ $\tan^2 x = \sec^2 x - 1$ $\tan 3y = \sqrt{sec^2 3y - 1} = \sqrt{x - 1}$	**M1**
$x = t^2$ $t = \sec 3y$ $\frac{dx}{dy} = 6x\sqrt{x - 1}$	**M1**
$\frac{dy}{dx} = \frac{1}{6x\sqrt{x-1}}$	**M1**

3c.

$\frac{d^2y}{dx^2} = \frac{(0)-1[(6x)(\frac{1}{2})(x-1)^{-\frac{1}{2}}(1)+(x-1)^{\frac{1}{2}}(6)]}{36x^2(x-1)}$	**M1**
$= \frac{\frac{-3}{(x-1)^{\frac{1}{2}}}-6(x-1)^{\frac{1}{2}}}{36x^2(x-1)} \times \frac{(x-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}}$	**M1**
$= \frac{-3x-6(x-1)}{36x^2(x-1)^{\frac{3}{2}}}$ $= \frac{6-9x}{36x^2(x-1)^{\frac{3}{2}}}$ $= \frac{3(2-3x)}{36x^2(x-1)^{\frac{3}{2}}}$	**M1**
$\frac{d^2y}{dx^2} = \frac{2-3x}{12x^2(x-1)^{\frac{3}{2}}}$	**M1**

4a.

$\frac{d}{dx}(\sec x) = \frac{d}{dx}(\frac{1}{\cos x}) = \frac{d}{dx}(\cos x)^{-1}$	**M1**
$= -(\cos x)^{-2}(-\sin x)$ $= \frac{1}{\cos^2 x} \times (\sin x)$ $= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$	**M1**
$= \sec x \tan x$	**M1**

4b.

$x = \sec 2y$ $\frac{dx}{dy} = (\sec 2y \tan 2y)(2)$	**M1**
$= 2 \sec 2y \tan 2y$	**M1**

4c.

$x = \sec 2y$ $\frac{dx}{dy} = 2 \sec 2y \tan 2y$ $\frac{dy}{dx} = \frac{1}{2 \sec 2y \tan 2y}$	**M1**
$1 + \tan^2 x = \sec^2 x$ $\tan x = \sqrt{sec^2 x - 1}$	**M1**
$\frac{dy}{dx} = \frac{1}{2 \sec 2y \sqrt{sec^2 2y - 1}}$	**M1**
Let $\sec 2y = x$ $\frac{dy}{dx} = \frac{1}{2x\sqrt{x^2-1}}$	**M1**

5a.

Quotient Rule: $u = \ln(x^2 + 1)$ $u' = \frac{2x}{x^2+1}$ $v = x$ $v' = 1$ $\frac{dy}{dx} = \frac{x\left(\frac{2x}{x^2+1}\right) - [\ln(x^2+1)](1)}{x^2}$	**M1**
$\frac{dy}{dx} = \frac{\left(\frac{2x^2}{x^2+1}\right) - [\ln(x^2+1)]}{x^2}$ $\frac{dy}{dx} = \frac{2x^2 - (x^2+1)\ln(x^2+1)}{x^2(x^2+1)}$	**M1**
$\frac{dy}{dx} = \frac{2x^2}{x^2(x^2+1)} - \frac{(x^2+1)\ln(x^2+1)}{x^2(x^2+1)}$	**M1**
$\frac{dy}{dx} = \frac{2}{(x^2+1)} - \frac{\ln(x^2+1)}{(x^2+1)}$	**M1**

5b.

$\frac{dx}{dy} = sec^2 y = 1 + \tan^2 y$	**M1**
$\frac{dx}{dy} = \frac{1+x^2}{1}$	**M1**
$\frac{dy}{dx} = \frac{1}{1+x^2}$	**M1**

6a.

When $y = \frac{\pi}{8}$ $x = 8(\frac{\pi}{8})(\tan\frac{\pi}{4}) = \pi$	**M1**

6b.

$\frac{dx}{dy} = (8y)[sec^2 2y](2) + (\tan 2y)(8)$	**M1 M1**
When $y = \frac{\pi}{8}$ $\frac{dx}{dy} = (8)(\frac{\pi}{8})[sec^2 2(\frac{\pi}{8})](2) + (\tan 2(\frac{\pi}{8}))(8)$ $\frac{dx}{dy} = 4\pi + 8$	**M1**
$\frac{dy}{dx} = \frac{1}{4\pi + 8}$	**M1**
Equation of tangent at P is: $y - \frac{\pi}{8} = \frac{1}{4\pi + 8}(x - \pi)$ $(4\pi + 8)y - \frac{\pi}{8}(4\pi + 8) = x - \pi$	**M1**
$(4\pi + 8)y = x - \pi + \frac{\pi}{8}(4\pi + 8)$	**M1**

7.

$\frac{dy}{dx} = \frac{1}{\frac{1}{3}x} \times \frac{1}{3} = \frac{1}{x}$	**M1**
When $x = 3$, $\frac{dy}{dx} = \frac{1}{3}$	**M1**

Therefore gradient of normal = -3	**M1**
Equation of normal at P is: $y - 0 = -3(x - 3)$	**M1**
$y = -3x + 9$	**M1**

8a.

$\frac{d}{dx}(16y^3 + 9x^2y - 54x) = 48y^2\frac{dy}{dx} + 9x^2(1)\frac{dy}{dx} + y(18x) - 54 = 0$	**M1 M1**
$(48y^2 + 9x^2)\frac{dy}{dx} = 54 - 18xy$	**M1**
$\frac{dy}{dx} = \frac{54-18xy}{48y^2+9x^2}$ $\frac{dy}{dx} = \frac{18(3-xy)}{3(16y^2+3x^2)}$	**M1**
$\frac{dy}{dx} = \frac{6(3-xy)}{16y^2+3x^2}$	**M1**

8b.

When $\frac{dy}{dx} = 0$ $6(3 - xy) = 0$	**M1**
$3 - xy = 0$ $xy = 3$ $x = \frac{3}{y}$	**M1**
$16y^3 + 9(\frac{3}{y})^2y - 54(\frac{3}{y}) = 0$	**M1**
$16y^3 + 9(\frac{9}{y^2}) - 54(\frac{3}{y}) = 0$ $16y^3 + \frac{81}{y} - \frac{162}{y} = 0$	**M1**
$16y^4 - 81 = 0$ $y = \pm\frac{3}{2}$	**M1**
When $y = \frac{3}{2}$, $x = 2$ When $y = -\frac{3}{2}$, $x = -2$	**M1** **M1**
Coordinates are: $(2, \frac{3}{2})$ and $(-2, -\frac{3}{2})$	

9.

$\frac{d}{dx}(2^x + y^2 = 2xy) = 2^x \ln 2 + 2y\frac{dy}{dx} = 2x(\frac{dy}{dx}) + y(2)$	**M1 M1** **M1**
$2^x \ln 2 - 2y = 2(x - y)\frac{dy}{dx}$	**M1**
$\frac{dy}{dx} = \frac{2^x \ln 2 - 2y}{2(x-y)}$	**M1**
At the point (3, 2) $\frac{dy}{dx} = \frac{2^3 \ln 2 - 2(2)}{2(3-2)}$	**M1**
$\frac{dy}{dx} = \frac{8 \ln 2 - 4}{2} = 4 \ln 2 - 2$	**M1**

10a.

$\frac{d}{dx}(ye^{-2x} = 2x + y^2) = y(-2e^{-2x}) + (e^{-2x})(1)\frac{dy}{dx} = 2 + 2y\frac{dy}{dx}$	**M1 M1** **M1**
$-2ye^{-2x} + e^{-2x}\frac{dy}{dx} = 2 + 2y\frac{dy}{dx}$	**M1**

$e^{-2x}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2 + 2ye^{-2x}$ $\frac{dy}{dx}(e^{-2x} - 2y) = 2(1 + ye^{-2x})$	
$\frac{dy}{dx} = \frac{2(1+ye^{-2x})}{e^{-2x}-2y}$	**M1**

10b.

$\frac{dy}{dx} = \frac{2(1+ye^{-2x})}{e^{-2x}-2y}$ When $x = 0$, $y = 1$ $\frac{dy}{dx} = \frac{2(1+(1)e^{-2(0)})}{e^{-2(0)}-2(1)} = -4$	**M1**
Gradient of normal = $\frac{1}{4}$	**M1**
Equation of normal at P: $y - 1 = \frac{1}{4}(x - 0)$ $4y - 4 = x$	**M1**
$x - 4y + 4 = 0$	**M1**

11.

$\frac{dV}{dt} = 0.48\pi - 0.6\pi h$	**M1**
$V = \pi(3)^2h = 9\pi h$	**M1**
$\frac{dV}{dh} = 9\pi$ $\frac{dh}{dV} = \frac{1}{9\pi}$	**M1**
$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $= \frac{1}{9\pi}(0.48\pi - 0.6\pi h)$ $= \frac{0.12\pi}{9\pi}(4 - 5h)$	**M1**
Therefore, $75\frac{dh}{dt} = 4 - 5h$	**M1**

12a.

$x = 7\cos t - \cos 7t$ $\frac{dx}{dt} = -7\sin t + 7\sin 7t$	**M1**
$y = 7\sin t - \sin 7t$ $\frac{dy}{dt} = 7\cos t - 7\cos 7t$	**M1**
$\frac{dy}{dx} = \frac{7\cos t - 7\cos 7t}{-7\sin t + 7\sin 7t}$	**M1**

12b.

when $t = \frac{\pi}{6}$ $\frac{dy}{dx} = \frac{7\cos(\frac{\pi}{6}) - 7\cos 7(\frac{\pi}{6})}{-7\sin(\frac{\pi}{6}) + 7\sin 7(\frac{\pi}{6})} = -\sqrt{3}$	**M1**
Gradient of normal = $\frac{1}{\sqrt{3}}$	**M1**
When $t = \frac{\pi}{6}$ $x = 7\cos\frac{\pi}{6} - \cos 7(\frac{\pi}{6}) = 4\sqrt{3}$	**M1**
when $t = \frac{\pi}{6}$ $y = 7\sin\frac{\pi}{6} - \sin 7(\frac{\pi}{6}) = 4$	**M1**
$y - 4 = \frac{1}{\sqrt{3}}(x - 4\sqrt{3})$ $3y = \sqrt{3}x$	**M1**