Algebra and Functions

Part 5: Composite Functions

AS Level	
Pt. 1: Index Laws & Surds	**Pt. 2**: Quadratic Functions
Pt. 3: Simultaneous Equations	**Pt. 4**: Graph Functions & Transformations

A-Level
Pt. 5: Composite Functions
Pt. 6: Modulus Functions
Pt. 7: Partial Fractions

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1. $g(x) = \frac{3x+8}{x-7}\ x \geq 9$

a. Find gg(11) **(2)**
b. State the range of $g(x)$ **(2)**
c. Find $g^{-1}(x)$ and state it's domain. **(3)**

2. The function f is defined by

$$f(x) = \frac{1}{5-x} \qquad x \in \mathbb{R},\ x \neq 5$$

a. Write down the range of $f(x)$ **(2)**
b. Find an expression for $f^{-1}(x)$ and state it's domain. **(3)**

$$g(x) = x^2 - 5 \qquad x \in \mathbb{R}$$

c. Solve $fg(x) = ¼$ **(3)**

3. The functions f and g are defined by

$$f : x \rightarrow \ln(3x-2),\ x \in \mathbb{R},\ x > \frac{2}{3}$$
$$g : x \rightarrow \frac{3}{x-2},\ x \in \mathbb{R},\ x \neq 2$$

a. Find the exact value of fg(3) **(2)**
b. Find an expression for $f^{-1}(x)$ and state it's domain **(4)**
c. Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram `

4. The function $g(x)$ is defined by $g(x) = x^2 - 8x + 7$, $\mathrm{x} \in \mathbb{R}$, $x > 4$. Find $g^{-1}(x)$ and state its domain and range. **(6)**

5. The functions f and g are definedby

$$f : x \rightarrow 1-5x^3 \qquad x \in \mathbb{R}$$
$$g : x \rightarrow \frac{2}{x} - 6, \qquad x > 0 \qquad x \in \mathbb{R}$$

a. Find the inverse function $f^{-1}(x)$. **(2)**
b. Show that the composite function of gf is

$$gf : x \rightarrow \frac{30x^3 - 4}{1-5x^3}$$

(4)
c. Solve $gf(x) = 0$ **(2)**

6. The function f is defined by,
$f : x \rightarrow \frac{x+1}{5y+10} \times \frac{y+2}{x^2+2x+1} \qquad x > 3$
a. Show that $f(x) = \frac{1}{5(x+1)}$ **(3)**
b. Find the range of $f(x)$ **(2)**
c. Find $f^{-1}(x)$. State the domain of this inverse function. **(3)**

7. The functions f and g are defined by

$$f : x \rightarrow 5x + \ln x$$
$$g : x \rightarrow e^{x^2}$$

a. Write down the range of $g(x)$ **(1)**

b. Show that the composite function $fg(x)$ is defined by:

$$fg : x \rightarrow x^2 + 5e^{x^2}$$

(2)

c. Write down the range of $fg(x)$ **(1)**

8. The function f is defined by

$$f : x \rightarrow x^2 + 1 \text{ for } x \geq 0$$

a. Define in a similar way the inverse function $f^{-1}(x)$. **(3)**

b. Solve the equation $ff(x) = \frac{185}{16}$ **(3)**

Mark Scheme

1a.

$g(11) = \frac{3(11)+8}{11-7} = \frac{41}{4}$ $g(\frac{41}{4}) = \frac{3(\frac{41}{4})+8}{\frac{41}{4}-7} = \frac{155}{13}$	**M1** **M1**

1b.

$g(9) = \frac{3(9)+8}{9-7} = \frac{35}{2}$ $g(\infty) \rightarrow 3$ Therefore range: $3 < g(x) \leq \frac{35}{2}$	**M1** **M1**

1c.

$y = \frac{3x+8}{x-7}$ $yx - 7y = 3x + 8$	**M1**
$x(y-3) = 8 + 7y$ $x = \frac{8+7y}{y-2}$	**M1**
$g^{-1}(x) = \frac{8+7x}{x-2}$	**M1**

2a.

$f(x) \in \mathbb{R}$ $f(x) \neq 0$	**M1** **M1**

2b.

$f(x) = \frac{1}{5-x} = y \rightarrow \frac{1}{5-y} = x$	**M1**
$1 = 5x - yx$ $y = \frac{5x-1}{x}$ $f^{-1}(x) = \frac{5x-1}{x}$	**M1**
Domain: $x \in \mathbb{R},\ x \neq 0$	**M1**

2c.

$fg(x) = \frac{1}{5-(x^2-5)}$ $\frac{1}{5-x^2+5}$	**M1**
$\frac{1}{-x^2+10} = \frac{1}{4}$ $4 = -x^2 + 10$	**M1**
$x^2 = 6$ $x = \pm\sqrt{6}$	**M1**

3a.

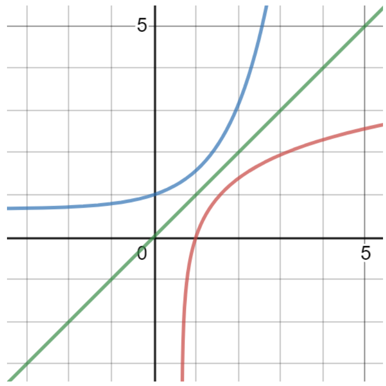
$g(3) = \frac{3}{3-2} = 3$	**M1**
$f(3) = \ln(9-2) = \ln 5$	**M1**

3b.

$y = \ln(3x - 2)$ $x = \ln(3y - 2)$	**M1**
$e^x = e^{\ln(3y-2)}$	**M1**
$e^x = 3y - 2$	**M1**
$y = \frac{e^x + 2}{3}$ $f^{-1}(x) = \frac{e^x + 2}{3}$	**M1**

3c.

Attempted reflection in $y = x$ **M1**
Red Line: $f(x)$ **M1**
Blue Line: $f^{-1}(x)$ **M1**



4.

$g(x) = x^2 - 8x + 7 = (x - 4)^2 - 9$	**M1**
$x = (y - 4)^2 - 9$	**M1**
$y - 4 = \sqrt{x + 9}$ $g^{-1}(x) = \sqrt{x + 9} + 4$	**M1** **M1**
Domain of $g(x)$ = Range of $g^{-1}(x) \rightarrow y > 4$	**M1**
Range of $g(x)$ = Domain of $g^{-1}(x) \rightarrow x > -9$	**M1**

5a.

$y = 1 - 5x^3 \rightarrow x = 1 - 5y^3$	**M1**
$5y^3 = 1 - x$ $y = \sqrt[3]{\frac{1-x}{5}}$ $f^{-1}(x) = \sqrt[3]{\frac{1-x}{5}}$	**M1**

5b.

$gf \rightarrow \frac{2}{1-5x^3} - 6$	**M1**
$\frac{2}{1-5x^3} - \frac{6(1-5x^3)}{1-5x^3}$	**M1**
$= \frac{2 - 6(1-5x^3)}{1-5x^3}$ $= \frac{2 - 6 + 30x^3}{1-5x^3}$	**M1**
$= \frac{30x^3 - 4}{1-5x^3}$	**M1**

5c.

$\frac{30x^3 - 4}{1-5x^3} = 0$ $30x^3 - 4 = 0$	**M1**
$30x^3 = 4$ $x = \sqrt[3]{\frac{2}{15}}$	**M1**

6a.

$\frac{x+1}{5y+10} \times \frac{y+2}{x^2+2x+1} = \frac{x+1}{5(y+2)} \times \frac{y+2}{(x+1)(x+1)}$	**M1 M1**
$\frac{1}{5} \times \frac{1}{(x+1)} = \frac{1}{5(x+1)}$	**M1**

6b.

$f(x) = \frac{1}{5(x+1)}, \quad x > 3$ $f(3) = \frac{1}{5(3+1)} = \frac{1}{20}$	**M1**
As x tend to ∞, $f(x)$ tends to 0.	**M1**
Range: $0 < f(x) < \frac{1}{20}$	**M1**

6c.

$f(x) = \frac{1}{5(x+1)} \to x = \frac{1}{5(y+1)}$	**M1**
$x = \frac{1}{5(y+1)}$ $5x(y+1) = 1$ $5xy + 5x = 1$ $y = \frac{1-5x}{5x}$	**M1**
Domain: $0 < x < \frac{1}{20}$	**M1**

7a.

$g(x) = e^{x^2}$ $x^2 \geq 0$, therefore, $e^{x^2} \geq 1$	**M1**

7b.

$fg : x \to 5e^{x^2} + \ln(e^{x^2})$	**M1**
$= 5e^{x^2} + x^2$	**M1**

7c.

Range of $5e^{x^2}$: $f(x) \geq 5$ Range of x^2: $f(x) \geq 0$ $gf(x) \geq 5$	**M1**

8a.

$y = x^2 + 1 \to x = y^2 + 1$	**M1**
$y = \sqrt{x-1}$	**M1**
$f^{-1} : x \to \sqrt{x-1}$	**M1**

8b.

$ff(x) \to (x^2+1)^2 + 1$	**M1**
$(x^2+1)^2 + 1 = \frac{185}{16}$ $(x^2+1)^2 = \frac{169}{16}$ $x^2 + 1 = \frac{13}{4}$ $x^2 = \frac{9}{4}$ $x = \pm\sqrt{\frac{3}{2}}$	**M1**
$x \geq 0$, therefore, $x = \sqrt{\frac{3}{2}}$	**M1**