Trignometry

Proof

AS-Level	A-Level
Sine and Cosine Rule	Small Angle Approximations Solving Trigonometry Equations Trignometric Proof

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1. Show that the equation $2\sin x = \frac{4\cos x - 1}{\tan x}$ can be expressed in the form $6\cos^2 x - \cos x - 2 = 0$ **(3)**

2. Show that the equation $3\sin^2 x + 7\sin x = \cos^2 x - 4$ can be written in the form $4\sin^2 x + 7\sin x + 3 = 0$ **(2)**

3. Prove that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$ **(4)**

4. Prove $\frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$ **(4)**

5. Prove that $\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \tan x$ **(3)**

6. Use the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$, to show that $\cos 2A = 1 - 2\sin^2 A$ **(2)**

7. Prove that $\cos x - \cos(x - \frac{\pi}{3}) \equiv \cos(x + \frac{\pi}{3})$ **(3)**

8. Prove that $\cos x \equiv 2\cos^2 \frac{x}{2} - 1$ **(2)**

9. Prove that $\frac{1 + \cos x}{1 - \cos x} \equiv \cot^2 \frac{x}{2}$ **(2)**

10. Use the indentities sin (A + B) and sin (A – B) to prove that $\sin P - \sin Q = 2\cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$ **(4)**

11. Prove that for all real values of x, $\cos(x + 30) + \sin x = \cos(x - 30)$ **(4)**

Mark Scheme

1.

$2\sin x \tan x = 4\cos x - 1$ $2\sin x \frac{\sin x}{\cos x} = 4\cos x - 1$ $2\sin^2 x = 4\cos^2 x - \cos x$	**M1**
$2(1 - \cos^2 x) = \cos x\,(4\cos x - 1)$	**M1**
$2 - 2\cos^2 x = 4\cos^2 x - \cos x$ $6\cos^2 x - \cos x - 2 = 0$	**M1**

2.

$3\sin^2 x + 7\sin x = \cos^2 x - 4$ $3\sin^2 x + 7\sin x = 1 - \sin^2 x - 4$	**M1**
$4\sin^2 x + 7\sin x + 3 = 0$	**M1**

3.

$\cos 4x = \cos 2(2x)$ $= 2\cos^2 2x - 1$	**M1**
$= 2(2\cos^2 x - 1)^2 - 1$	**M1**
$= 2(2\cos^2 x - 1)(2\cos^2 x - 1) - 1$ $= 2(4\cos^4 x - 2\cos^2 x - 2\cos^2 x + 1) - 1$ $= 2(4\cos^4 x - 4\cos^2 x + 1) - 1$	**M1**
$= 8\cos^4 x - 8\cos^2 x + 2 - 1$ $= 8\cos^4 x - 8\cos^2 x + 1$	**M1**

4.

$\frac{\sin x}{1+\cos x} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2cos^2\frac{x}{2}-1}$	**M1**
$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2cos^2\frac{x}{2}}$	**M1**
$= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$	**M1**
$= \tan\frac{x}{2}$	

5.

$\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \frac{1-\cos 2x}{\sin 2x}$	**M1**
$= \frac{1-(1-2sin^2 x)}{2\sin x\cos x}$ $= \frac{2sin^2 x}{2\sin x\cos x}$	**M1**
$= \frac{\sin x}{\cos x}$ $= \tan x$	**M1**

6.

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ Let $B = A$ $\cos(A + A) = \cos A\cos A - \sin A\sin A$ $\cos 2A = \cos^2 A - \sin^2 A$	**M1**
$= 1 - \sin^2 A - \sin^2 A$ $= 1 - 2\sin^2 A$	**M1**

7.

(LHS) = cos x – (cos x cos $\frac{\pi}{3}$ + sin x sin $\frac{\pi}{3}$)	**M1**
= cos x - $\frac{1}{2}$cos $x - \frac{\sqrt{3}}{2}$ sin x = $\frac{1}{2}$cos $x - \frac{\sqrt{3}}{2}$ sin x	**M1**
= cos x cos $\frac{\pi}{3}$ – sin x sin $\frac{\pi}{3}$ = cos (x + $\frac{\pi}{3}$)	

8.

cos 2A ≡ 2cos^2A – 1 let $A = \frac{x}{2}$	**M1**
cos x ≡ 2 cos$^2\frac{x}{2}$ - 1	**M1**

9.

(LHS) = $\frac{1+(2cos^2\frac{x}{2})}{2sin^2\frac{x}{2}}$	**M1**
= cot$^2\,\frac{x}{2}$	**M1**

10.

sin(A + B) = sin A cos B = cos A sin B sin (A – B) = sin A cos B – cos A sin B	**M1**
sin(A + B) – sin(A – B) = 2 cos A sin B let P = A + B (1) Q = A – B (2)	**M1**
(1) + (2) ⟶ 2A = P + Q ⟶ $A = \frac{P+Q}{2}$ (1) - (2) ⟶ 2B = P - Q ⟶ $B = \frac{P-Q}{2}$	**M1**
sin P –sin Q = 2 cos $\frac{P+Q}{2}$ sin $\frac{P-Q}{2}$	**M1**

11.

(LHS) = cos x cos30 – sin x sin30 + sin x	**M1**
= $\frac{\sqrt{3}}{2}$cos x − ½ sin x + sin x = $\frac{\sqrt{3}}{2}$cos x + $\frac{1}{2}$sin x	**M1**
= cos x cos30 + sin x sin30	**M1**
= cos (x – 30)^o (RHS)	**M1**

Trignometry

Solving Trig Equations

AS-Level	**A-Level**	
Sine and Cosine Rule	Small Angle Approximations	Solving Trigonometry Equations
	Trignometric Proof	

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1. Solve the equation sec x = 5, giving all the values of x in the interval $0 \leq x \leq 2\pi$ in radians to two decimal places. **(3)**
b. Show that the equation $\tan^2 x = 3\sec x + 9$ can be written as $\sec^2 x - 3\sec x - 10 = 0$ **(2)**
c. Solve the equation $\tan^2 x = 3\sec x - 10 = 0$ **(4)**

2. f(x) = 12 cos x – 4 sin x
Given that f(x) = R cos($x + \alpha$), where R ≥ 0 and $0 \leq \alpha \leq 90$.
a. Find the value of R and the value of α. **(4)**
b. Hence solve the equation, 12cos x – 4 sin x = 7, for $0 \leq x \leq 360$, giving your answers to one decimal place. **(5)**
c. Write down the minimum value of 12cos x – 4sin x. **(1)**
d. Find to two decimal places, the smallest postive value of x for which this minimum value occurs **(2)**

3a. Prove that $\operatorname{cosec}^4\theta - \cot^4\theta = \operatorname{cosec}^2\theta + \cot^2\theta$ **(2)**
b. Solve for $90^{O} < \theta\ 180^{O}$, $\operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta$ **(5)**

4. Solve $\operatorname{cosec}^2 2x - \cot 2x = 1$ for $0 \leq x \leq 180$. **(7)**

5a. Prove that $\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \tan x$ **(4)**
b. Hence, or otherwise, show that tan 15 = 2 - $\sqrt{3}$ **(3)**
c. Solve cosec 4x – cot 4x = 1, for 0 < x < 360. **(5)**

6. Solve for $2\cot^2 3x = 7\operatorname{cosec} 3x - 5$, $0 \leq x \leq 180$ **(10)**

7a. Starting from the formulae for sin (A + B) and cos (A + B), prove that tan (A + B) = $\frac{\tan A + \tan B}{1 - \tan A tan B}$ **(4)**
b. Deduce that $\tan\left(x + \frac{\pi}{6}\right) = \frac{1 + \sqrt{3}\tan x}{\sqrt{3} - \tan x}$ **(3)**
c. Hence, or otherwise, solve $1 + \sqrt{3}\tan x = (\sqrt{3} - \tan x)\tan(\pi - x)$ **(6)**

8a. Express 3 cos θ + 4 sin θ in the form R cos (θ − α), where R > 0 and 0 < α < $\frac{\pi}{2}$. **(4)**
b. Given that the function f is defined by f(θ) ≡ 1 − 3 cos 2θ − 4 sin 2θ, $\theta \in \mathbb{R}$, $0 \leq \theta \leq \pi$, state the range of f. **(2)**
c. Solve the equation f(θ) = 0. (6) **(4)**
d. Find the coordinates of the turning points of the curve with equation y = $\frac{2}{3\cos x + 4\sin x}$ + in the interval [0, 2π]. **(3)**

9a. Express tan 2x in terms of tan x and hence solve, for 0 < x < 180, the equation tan 2x tan x = 8. **(5)**
b. Given that β is the acute angle such that $\beta = \frac{6}{7}$, find the exact value of
i. cosec β **(1)**
ii. $\cot^2\beta$ **(1)**

10a. Use the identity for cos(A + B) to prove that $4\cos(x + 60)\cos(x + 30) = \sqrt{3} - 2\sin 2x$. **(4)**

b. Hence, find the exact value of 4cos(82.5)cos(52.5). **(2)**

c. Solve for $0 < x < 90$, the equation 4cos(82.5)cos(52.5) **(2)**

d. Given that there are no values of θ which satisfy the equation $4\cos(x + 60)\cos(x + 30) = k$, determine the set of values of the constant k. **(2)**

Mark Scheme

1a.

Sec x = 5 $\frac{1}{\cos x} = \frac{1}{5}$	**M1**
Cos x = 0.2	**M1**
x = 1.37, 4.91	**M1**

1b.

$\tan^2 x = 3\sec x + 9$ $\sec^2 x - 1 = 3\sec x + 9$	**M1**
$\sec^2 x - 3\sec x - 10 = 0$	**M1**

1c.

(sec x – 5)(sec x + 2) = 0	**M1**
Sec x = 5, -2	**M1**
cos x = 0.2, -0.5	**M1**
x = 1.37, 4.91, 2.09, 4.19	**M1**

2a.

R cos α = 12 R sin α = 4	**M1**
$R^2 = 12^2 + 4^2$ R = $\sqrt{160}$	**M1**
tan $\propto = \frac{4}{12}$	**M1**
$\propto$ = 18.43	**M1**

2b.

cos (x + 18.43) = $\frac{7}{\sqrt{160}}$	**M1**
$x + 18.43 = 56.4^{O}$	**M1**
$360 - 56.4 = 303.60^{O}$	**M1**
$x = 38.0^{O}$	**M1**
$x = 285.2^{O}$	**M1**

2c.

Minimum value is -$\sqrt{160}$	**M1**

2d.

cos(x + 18.43) = -1	**M1**
$x = 161.57^{O}$	**M1**

3a.

$\text{cosec}^4\theta - \cot^4\theta = (\text{cosec}^2\theta - \cot^2\theta)(\text{cosec}^2\theta + \cot^2\theta)$ $1 + \cot^2\theta = \text{cosec}^2\theta$	**M1**
$= (\text{cosec}^2\theta - \cot^2\theta)$	**M1**

Jethwa Maths

3b.

$\text{cosec}^2\theta - \cot^2\theta = 2 - \cot\theta$	**M1**
$1 + \cot^2\theta + cot^2\theta = 2 - \cot\theta$ $2\cot^2\theta + \cot\theta = 0$	**M1**
$(2\cot\theta - 1)(cot\,\theta + 1) = 0$	**M1**
$\cot\theta = \frac{1}{2}$ $\cot\theta = -1$	**M1**
$\theta = 135$	**M1**

4.

$\text{cosec}^2 2x - \cot 2x = 1$ (using $\text{cosec}^2 2x = 1 + \cot^2 2x$)	**M1**
$1 + \cot^2 2x - \cot 2x = 1$ $\cot^2 2x - \cot 2x = 0$	**M1**
$\cot 2x(\cot 2x - 1) = 0$	**M1**
$\cot 2x = 0$ $\tan 2x \rightarrow \infty$	**M1**
$x = 90, 270$	**M1**
$\cot 2x = 1$ $\tan 2x = 1$	**M1**
$2x = 45, 225$ $x = 22.5, 112.5$	**M1**
$x = 22.5, 45, 112.5, 135$	

5a.

$\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \frac{1 - cox\,2x}{\sin 2x}$	**M1**
$= \frac{2sin^2 x}{2\,sinx\cos x}$	**M1**
$= \frac{\sin x}{\cos x}$	**M1**
$= \tan x$	**M1**

5b.

$\tan 15 = \frac{1}{\sin 30} - \frac{\cos 30}{\sin 30}$	**M1**
$\tan 15 = \frac{1}{0.5} - \frac{\frac{\sqrt{3}}{2}}{0.5}$	**M1**
$= 2 - \sqrt{3}$	**M1**

5c.

$\tan 2x = 1$	**M1**
$2x = 45$ $x = 22.5, 202{,}5$	**M1 M1**
$2x = 45 + 180$ $x = 112.5, 292.5$	**M1 M1**
$x = 22.5, 112.5, 202.5, 292.5$	

6.

$\cot^2(3x) = \text{cosec}^2(3x) - 1$	**M1**
$2\cot^2(3x) = 7\,\text{cosec}(3x) - 5$	**M1**
$2\text{cosec}^2(3x) - 7\text{cosec}(3x) + 3 = 0$ $(2\text{cosec}\,3x - 1)(\text{cosec}\,3x - 3) = 0$	**M1**
$2\text{cosec}\,3x - 1 \rightarrow$ no solution	**M1**
$\text{cosec}\,3x - 3 \rightarrow \text{cosec}\,3x = 3$	**M1**
$\frac{1}{\sin 3x} = \frac{1}{3}$	**M1**

$x = 6.5$ $x = 53.5$	**M1 M1**
$x = 126.5$ $x = 173.5$	**M1 M1**

7a.

$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$	**M1**
$= \frac{\frac{sinA}{cosA}+\frac{sinB}{cosB}}{1-\frac{sinAsinB}{cosAcosB}}$	**M1**
(divide by ÷ cosAcosB)	**M1**
$= \frac{tanA+tanB}{1-tanAtanB}$	**M1**

7b.

$\tan(x + \frac{\pi}{6}) = \frac{\tan x+tan\frac{\pi}{6}}{1-tanxtan\frac{\pi}{6}}$	**M1**
$= \frac{\tan x+\frac{1}{\sqrt{3}}}{1-tanx\frac{1}{\sqrt{3}}}$	**M1**
$= \frac{\sqrt{3}\tan x+1}{\sqrt{3}-\tan x}$	**M1**

7c.

$\tan(x + \frac{\pi}{6}) = \tan(\pi - x)$	**M1**
$x + \frac{\pi}{6} = \pi - x$	**M1**
$x = \frac{5}{12}\pi$	**M1**
$\tan(x + \frac{\pi}{6}) = \tan(2\pi - x)$	**M1**
$x + \frac{\pi}{6} = 2\pi - x$	**M1**
$x = \frac{11}{12}x$	**M1**

8a.

$3\cos\theta + 4\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$	**M1**
$R\cos\alpha = 3$ $R\sin\alpha = 4$	**M1**
$R = \sqrt{9+16} = 5$	**M1**
$\tan\alpha = \frac{4}{3}$ $\alpha = 0.927$	**M1**
$3\cos\theta + 4\sin\theta = 5\cos(\theta = 0.927)$	

8b.

$-4 \leq f(\theta) \leq 6$	**M1**

8c.

$1 - 5\cos(2\theta - 0.9273) = 0$	**M1**
$\cos(2\theta - 0.9273) = \frac{1}{5}$	**M1**
$2\theta - 0.9273 = 1.3694$ $2\theta = 2.2967$ $\theta = 1.15$	**M1**
$2\theta - 0.9273 = 2\pi - 1.3694$ $2\theta = 5.8410$ $\theta = 2.92$	**M1**

8d.

y = $\frac{2}{5\cos(x-0.9273)}$	**M1**
Turning Point: $y = \frac{2}{5}$ when $x - 0.9273 = 0$ $x = 0.9273$	**M1**
$y = -\frac{2}{5}$ when $x - 0.9273 = \pi$ $x = 4.07$	**M1**
Turning Points: $(0.93, \frac{2}{5})$, $(4.07, -\frac{2}{5})$	

9a.

$\tan 2x = \frac{2\tan x}{1 - tan^2 x}$	**M1**
$\tan^2 x = \frac{4}{5}$	**M1**
$\tan x = \pm\sqrt{\frac{4}{5}}$	**M1**
$x = 41.8$	**M1**
$x = 138.2$	**M1**

9bi.

$\frac{7}{6}$	**M1**

9bii.

$\text{cosec}^2 x = 1 + \cot^2 x$ $\cot x = \frac{13}{36}$	**M1**

10a.

$\cos(x + 60) = \cos x \cos 60 - \sin x \sin 60$ $\cos(x + 30) = \cos x \cos 30 - \sin x \sin 30$	**M1**
$\cos(x + 60)\cos(x + 30) = (\cos x \cos 60 - \sin x \sin 60)(\cos x \cos 30 - \sin x \sin 30)$ $= \cos^2 x \cos 60 \cos 30 + \sin^2 x \sin 60 \sin 30 - \cos x \sin x \cos 60 \sin 30 - \sin x \cos x \sin 60 \cos 30$ $= \cos^2 x \frac{\sqrt{3}}{4} + \sin^2 x \frac{\sqrt{3}}{4} - \frac{1}{4}\cos x \sin x - \frac{3}{4}\cos x \sin x$ $= \frac{\sqrt{3}}{4}(\cos^2 x + \sin^2 x) - \cos x \sin x$ $= \frac{\sqrt{3}}{4} - \sin x \cos x$	**M1**
$4\cos(x + 60)\cos(x + 30) = 4(\frac{\sqrt{3}}{4} - \cos x \sin x) = \sqrt{3} - 4\cos x \sin x$ $4\cos x \sin x = 2\sin 2x$	**M1**
$= \sqrt{3} - 2\sin 2x$	**M1**

10b.

$x = 22.5$	**M1**
$4\cos(82.5)\cos(52.5) = \sqrt{3} - 2\sin 2(22.5) = \sqrt{3} - \sqrt{2}$	**M1**

10c.

$x = 10.7$	**M1**
$x = 79.3$	**M1**

10d.

Critical values of sin 2x = -1 and 1	**M1**
Therefore, $k > \sqrt{3} + 2$ $k < \sqrt{3} - 2$	**M1**

Small Angle Approximations

AS-Level	A-Level
Sine and Cosine Rule	Small Angle Approximations, Solving Trigonometry Equations, Trignometric Proof

1. Show that, for a small angle Ø, where Ø is measured is in radians, **(3)**

$$1 + \cos Ø - 3cos^2 Ø \approx -1 + \frac{5}{2}Ø^2$$

2. Use small angle approximations to estimate the solution of the equation $\frac{cos\frac{1}{2}Ø}{1+sinØ} = 0.825$, if Ø is small enough to neglect terms in $Ø^3$ or above. **(4)**

3a. Given that Ø is small, use the small angle approximation of cos Ø to show that

$$4 \cos Ø + \cos^2(2Ø) \approx 5 - 6Ø^2 + 4Ø^4$$ **(3)**

b. Hence find an approximation of 4 cos Ø + $\cos^2(2Ø)$ when Ø = 3° **(2)**

c. Calcualte the percentage error in your approximation **(1)**

Mark Scheme

1.

$1 + (1 - \frac{\emptyset^2}{2}) - (3[1 - \frac{\emptyset^2}{2}]^2)$	**M1**
$= 1 + (1 - \frac{\emptyset^2}{2}) - 3(1 - \emptyset^2 + \frac{\emptyset^2}{4})$ $= 1 + 1 - \frac{\emptyset^2}{2} - 3 + 3\emptyset^2 - \frac{3\emptyset^2}{4}$	**M1**
As $\emptyset$ is very small, we can ignore the higher order terms, therefore, $1 + \cos\emptyset - 3cos^2\emptyset \approx -1 + \frac{5}{2}\emptyset^2$	**M1**

2.

$\frac{1-\frac{1}{8}\emptyset^2}{1+\emptyset} = 0.825$	**M1**
$0.125\emptyset^2 + 0.825\emptyset - 0.175 = 0$	**M1**
$\emptyset = 0.206$ or -6.81	**M1**
As $\emptyset$ must be small, $\emptyset = 0.206$	**M1**

3a.

$4\cos\emptyset + \cos^2(2\emptyset) \approx 4(1 - \frac{\emptyset^2}{2}) + (1 - \frac{(2\emptyset)^2}{2})^2$	**M1**
$\approx 4 - 2\emptyset^2 + (1 - 2\emptyset)^2$	**M1**
$\approx 4 - 2\emptyset^2 + 1 - 4\emptyset^2 + 4\emptyset^4$ $\approx 5 - 6\emptyset^2 + 4\emptyset^4$	**M1**

3b.

$3^0 = \frac{3}{180} \times \pi^2 = \frac{1}{60}\pi^2$ $5 - 6(\frac{1}{60}\pi^2) + 4(\frac{1}{60}\pi^2)^4$ $= 4.983580724$	**M1**

3c.

$4\cos(\frac{1}{60}\pi) + \cos^2(\frac{1}{30}\pi) = 4.983591939$	**M1**
% error $= \frac{4.983591939 - 4.983580724}{4.983591939} \times 100$	**M1**
% error $= 2.25 \times 10^{-4}$ % (to 3 significant figures)	**M1**