A-Level Unit Test: Exponentials and Logarithms

Exponentials and Logarithms

Jethwa Maths

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1. Solve the equation $2^{2x+5} - 7(2^x) = 0$ **(4)**

2a. $2\log(x + a) = \log(16a^6)$, where a is a positive constant. Find x in terms of a, giving your answer in its simplest form. **(3)**
b. $\log_3(9y + b) - \log_3(2y - b) = 2$, where b is a positive constant. Find y in terms of b, giving your answer in its simplest form. **(4)**

3. Given that $\log_3 x = a$, find in terms of a, giving each answer in its simplest form
a. $\log_3(9x)$ **(2)**
b. $\log_3\frac{x^5}{81}$ **(3)**
c. Solve for x, $\log_3(9x) + \log_3\frac{x^5}{81} = 3$ **(4)**

4a. Simplify fully: $\frac{2x^2+9x-5}{x^2+2x-15}$ **(2)**
b. Given that $\ln(2x^2 + 9x - 5) = 1 + \ln(x^2 + 2x - 15)$, find x in terms of e. **(4)**

5. On 1 January 1990, a sculpture was values at £80. When the sculpture was sold on 1 January 1956, its value was £5000. The value of £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1990 and A and K are constants.
a. Write down the value of A **(1)**
b. Show that $k \approx 1.07664$ **(3)**
c. Use this model to show that the value of the sculpture on 1 January 2006 will be greater than £200 000 **(2)**
d. Find the year in which the value of the sculpture will first exceed £800,000 **(2)**

6. A biologist is researching the growth of a certain species of hamster. She proposes that the length x cm of a hamster t days after its birth is given by,

$$x = 15 - 12e^{-\frac{t}{14}}$$

a. Use this model to find the length of a hamster when it is born **(1)**
b. Find the length of a hamster after 14 days, giving your answer to three significant figures. **(2)**
c. Show that the time for a hamster to grow to 10cm in length is given by $t = 14\ln(\frac{a}{b})$, where a and b are integers **(2)**
d. Find this time to the nearest day **(1)**

7. A particular species of orchid is being studied. The population p at time t years after the study started is assumed to be

$$p = \frac{2800ae^{0.2t}}{1+ae^{0.2t}}, \text{ where a is a constant}$$

Given that there were 300 orchids when the study starter,
a. Show that $a = 0.12$ **(3)**
b. Use the equation with $a = 0.12$ to predict the number of years before the population of orchids reaches 1850 **(4)**
c. Show that $p = \frac{336}{0.12+e^{-0.2t}}$ **(1)**
d. Hence show that the population cannot exceed 2800. **(2)**

8. Find the exact solutions to the equations,
a. $\ln x + \ln 3 + \ln 6$ **(2)**
b. $e^x + 3e^{-x} = 4$ **(4)**

9. The amount of a certain type of drug in the bloodstream t hours after it has been taken is given by the formula, $x = De^{-\frac{1}{8}t}$,

Where x is the amount of the drug in the bloodstream in milligrams and D is the dose given in milligrams.

A dose of 10mg of the drug is given.

a. Find the amount of the drug in the bloodstream 5 hours after the dose is given. Give your answer in mg to 3 decimal places. **(2)**

A second dose of 10mg is given after 5 years.

b. Show that the amount of the drug in the bloodstream 1 hour after the second dose is 13.549 mg to 3 decimal places. **(2)**

No more doses of the drug are given. At time T hours after the second dose is given, the amount of the drug in the bloodstream is 3 mg.

c. Find the value of T **(3)**

10. The radioactive decay of a substance is given by

$$R = -1000e^{-ct},\ t \geq 0.$$

where R is the number of atoms at time t years and c is a positive constant.
a. Find the number of atoms when the substance started to decay. **(1)**

It takes 5730 years for half of the substance to decay.

b. Find the value of c to 3 significant figures. **(4)**
c. Calculate the number of atoms that will be left when t = 22 920. **(2)**
d. In the space provided on page 13, sketch the graph of R against t. **(2)**

11. Find the exact solutions to the equations
a. $\ln(3x - 7) = 5$ **(3)**
b. $3^x e^{7x+2} = 15$ **(5)**

The function f and g are defined by
$f(x) = e^{2x} + 3$
$g(x) = \ln(x - 1)$

c. Find f^{-1} and state its domain **(4)**
d. Find fg and state its range **(3)**

Total marks: 87

Mark Scheme

1.

$2^{2x+5} - 7(2^x) = 0$ $2^{2x+5} = 7(2^x)$	**M1**
$\log_2(2^{2x+5}) = \log_2(7(2x))$	**M1**
$2x + 5 = \log_2 7 + \log_2 2^x$ $2x + 5 = \log_2 7 + x$	**M1**
$x = \log_2 7 - 5$ $x = -2.19$	**M1**

2a.

$\log(x + a) = \frac{1}{2}\log(16a^6)$ $\log(x + a) = \log\sqrt{16a^6}$	**M1**
$x + a = \sqrt{16a^6}$ $x + a = 4a^3$	**M1**
$x = 4a^3 - a$	**M1**

2b.

$\log_3(9y + b) - \log_3(2y - b) = 2$ $\log_3(\frac{9y+b}{2y-b}) = 2$	**M1**
$\frac{9y+b}{2y-b} = 3^2$	**M1**
$9y + b = 18y - 9b$ $9y = 10b$	**M1**
$y = \frac{10}{9}b$	**M1**

3a.

$\log_3(9x) = \log_3(9) + \log_3(x)$	**M1**
$= 2 + \log_3 x$ $= 2 + a$	**M1**

3b.

$\log_3\frac{x^5}{81} = \log_3(x^5) - \log_3(81)$	**M1**
$= 5\log_3(x) - \log_3(81)$ $= 5\log_3(x) - 4$	**M1**
$= 5a - 4$	**M1**

3c.

$2 + a + 5a - 4 = 3$ $6a = 5$	**M1**
$a = \frac{5}{6}$	**M1**
$\frac{5}{6} = \log_3(x)$ $3^{\frac{5}{6}} = x$	**M1**
$x = 2.498$	**M1**

4a.

$\frac{2x^2+9x-5}{x^2+2x-15} = \frac{(2x-1)(x+5)}{(x-3)(x+5)}$	**M1**
$= \frac{2x-1}{x-3}$	**M1**

4b.

$\ln\frac{2x^2+9x-5}{x^2+2x-15} = 1$	**M1**
$e = \frac{2x^2+9x-5}{x^2+2x-15} = \frac{2x-1}{x-3}$	**M1**
$ex - 3e = 2x - 1$ $ex - 2x = 3e - 1$ $x(e - 2) = 3e - 1$	**M1**
$x = \frac{3e-1}{e-2}$	**M1**

5a.

$A = 80$	**M1**

5b.

$5000 = 80 \text{ x } k^{56}$ $k = \sqrt[56]{\frac{5000}{80}} = 1.07664$	**M1**

5c.

$V = 80 \text{ x } k^{106}$ $= 200707$	**M1**

5d.

$\ln 10000 = \ln k^t$ $t = \frac{\ln 10000}{\ln k} = 124.7$ $= 2024$	**M1**

6a.

$t = 0, x = 3$	**M1**

6b.

$t = 14$ $x = 15 - 12e^{-1}$	**M1**
$x = 10.6$	**M1**

6c.

$-5 = -12e^{-\frac{t}{14}}$ $\ln(\frac{5}{12}) = -\frac{t}{14}$	**M1**
$t = 14 \ln (\frac{12}{5})$	**M1**

6d.

$t = 12.256 \approx 12$ days	**M1**

7a.

$p = 300$ at $t = 0$ $300 = \frac{2800a}{1+a}$	**M1**
$300 = 2500a$	**M1**
$a = 0.12$	**M1**

7b.

$1850 = \frac{2800(0.12)e^{0.2t}}{1+0.12e^{0.2t}}$	**M1**
$e^{0.2t} = 16.2$	**M1**
$0.2t = \ln k$	**M1**
$t = 14$	**M1**

7c.

$p(0.12) = \frac{2800(0.12)e^{0.2t}}{1+(0.12)e^{0.2t}}$ $p = \frac{336e^{0.2t}}{1+(0.12)e^{0.2t}}$	**M1**
Divide numerator and denominator by $e^{0.2t}$ $p = \frac{336}{e^{-0.2t}+0.12}$	**M1**

7d.

$t \to \infty$ $e^{-0.2t} \to 0$	**M1**
$p \to \frac{336}{0.12} = 2800$	**M1**

8a.

$3x = 6$	**M1**
$x = 2$	**M1**

8b.

$e^x + 3e^{-x} = 4$ $e^x + \frac{3}{e^x} = 4$	**M1**
$(e^x)^2 + 3 = 4e^x$ $(e^x)^2 - 4e^x + 3 = 0$ $(e^x - 3)(e^x - 1) = 0$	**M1**
$e^x - 3 = 0$ $e^x = 3$ $x = \ln 3$	**M1**
$e^x - 1 = 0$ $e^x = 1$ $x = \ln 1 = 0$	**M1**

9a.

$D = 10, t = 5$ $x = 10e^{-\frac{1}{8}\times 5}$	**M1**
$x = 5.353$	**M1**

9b.

$D = 10 + 10e^{-\frac{5}{8}}, t = 1$ $x = 15.3526\ldots \times e^{-\frac{1}{8}}$	**M1**
$x = 13.549$	**M1**

9c.

$15.3526..\ e^{-\frac{1}{8}T} = 3$	**M1**
$e^{-\frac{1}{8}T} = \frac{3}{15.3526} = 0.1954$	**M1**
$-\frac{1}{8}T = \ln 0.1954$ $T = 13.06$	**M1**

10a.

1000	**M1**

10b.

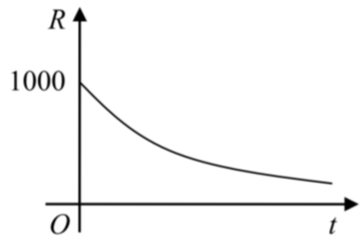
$1000e^{-5730c} = 500$ $e^{-5730c} = ½$	**M1**
$-5730c = \ln\frac{1}{2}$ $c = 0.000121$	**M1**

10c.

$R = 1000\ e^{-22920c} = 62.5$	**M1**

10d.

Shape **M1**
1000 **M1**



11a.

$\ln(3x - 7) = 5$ $e^{\ln(3x-7)} = e^5$	**M1**
$3x - 7 = e^5$	**M1**
$x = \frac{e^5+7}{3}$	**M1**

11b.

$3^x\ e^{7x+2} = 15$ $\ln(3^x\ e^{7x+2}) = \ln 15$	**M1**
$\ln 3^x + \ln e^{7x+2} = \ln 15$	**M1**
$\ln 3^x + 7x + 2 = \ln 15$ $x \ln 3 + 7x + 2 = \ln 15$	**M1**
$x(\ln 3 + 7) = -2 + \ln 15$	**M1**
$x = \frac{-2+\ln 15}{7+\ln 3}$	**M1**

11c.

$f(x) = e^{2x} + 3$ $y = e^{2x} + 3$	**M1**
$y - 3 = e^{2x}$ $\ln(y - 3) = 2x$	**M1**
$x = ½ \ln(y - 3)$ $f^{-1}(x) = ½ \ln(x - 3)$	**M1**
$f^{-1}(x)$ Domain: $x > 3$	**M1**

11d.

$g(x) = \ln(x - 1)$, $x > 1$	**M1**
$fg(x) = e^{2\ln(x-1)} + 3$	**M1**
$fg(x)$: Range $y > 3$	**M1**