

Exponentials and Logarithms



1. Express $10^3 = 1000$ in the form $\log_a b = c$ (1)
2. Express $\log_a 4 + \log_a 7$ in the form $\log_a n$ (1)
3. Express $4\log_a \frac{1}{\sqrt{x}}$ in the form $p\log_a x$ (1)
4. Given that $y = \log_q 8$, express $\log_q 64$ in terms of y (1)
5. Solve $\log_4(0.5x + 1) = 3.2$, giving your answer to 3 significant figures. (2)
6. Solve $\log_3 x + \log_3 5 = \log_3(2x + 3)$ (2)
7. Solve the simultaneous equations (5)

$$\begin{aligned} \log_2 x &= 3 - 2\log_2 y \\ \log_y 32 &= \frac{5}{2} \end{aligned}$$
8. Solve $10^x = 14$ to 2 decimal places. (1)
9. Solve $16 - 3^{4+x} = 0$ (3)
10. Solve $3(16^x) - 4^{x+2} = 0$ (3)
11. Solve $3^{2x+1} + 15 = 2(3^{x+2})$ (4)
12. Sketch the following two curves $y = 2^x$ and $y = 2^{x+3}$ on the same diagrams (3)
13. Given that $a = \log_{10} 2$ and $b = \log_{10} 3$, find an expression in terms of a and b for $\log_{10} 1.5$ (2)
14. Given that $2\log_2(x + 15) - \log_2 x = 6$
 - a. Show that $x^2 - 34x + 225 = 0$ (4)
 - b. Hence, or otherwise, solve the equation, $2\log_2(x + 15) - \log_2 x = 6$ (2)
- 15a. Find, to 3 significant figures, the values of x for which $5^x = 7$. (2)
- b. Solve the equation $5^{2x} - 12(5^x) + 35 = 0$ (4)
16. The mass, m grams, of a leaf t days after it has been picked from a tree is given by (1)

$$m = pe^{-kt}$$

Where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

 - a. Write down the value of p (1)
 - b. Show that $k = \frac{1}{4} \ln 3$ (4)
17. A heated metal ball is dropped into a liquid. As the ball cools, its temperature $T^\circ\text{C}$, t minutes after it enters the liquid is given by, (1)

$$T = 400e^{-0.05t} + 25, t \geq 0$$
 - a. Find the temperature of the ball as it enters the liquid. (1)
 - b. Find the value of t for which $T = 300$, giving your answer to 3 significant figures. (3)

18. The temperature of a mug of coffee at time t can be modelled by the equation

$T(t) = T_R + (90 - T_R)e^{-\frac{1}{20}t}$ where $T(t)$ is the temperature, in $^{\circ}\text{C}$, of the coffee at time t minutes after the coffee was poured into the mug and T_R is the room temperature in $^{\circ}\text{C}$.

Using the equation for this model,

a. Explain why the initial temperature of the coffee is independent of the initial room temperature. (2)

b. Calculate the temperature of the coffee after 10 minutes if its room temperature is 20°C . (2)

19. Given that $\log_3(3b + 1) - \log_3(a - 2) = -1$, express b in terms of a (3)

20. Joan brings a cup of hot tea into a room and places the cup on a table. At time t minutes after Joan places the cup on the table, the temperature, $\theta^{\circ}\text{C}$, of the tea is modelled by the equation

$$\theta = 20 + Ae^{-kt}$$

where A and k are positive constants. Given that the initial temperature of the tea was 90°C .

a. Find the value of A (2)

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C .

b. Show that $k = \frac{1}{5}\ln 2$ (3)

21. The area, $A \text{ mm}^2$, of a bacterial culture growing in milk, t hours after midday, is given by

$$A = 20e^{1.5t}, t \geq 0$$

a. Write down the area of the culture at midday. (1)

b. Find the time at which the area of the culture is twice its area at midday. Give your answer to the nearest minute. (4)

22. The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$$

Where V is the values of the car in pounds (£) and t is the age in yeats.

a. Find the value of the car when $t = 0$ (1)

b. Calculate the exact values of t when $V = 9500$ (4)

23. The population of a town is being studied. The population P , at time t years from the start of the study, is assumed to be,

$$P = \frac{8000}{1+7e^{-kt}}, t \geq 0$$

Where k is a postive constant.

a. Use the given equation to find the population at the start of the study. (2)

b. Find a value for the expected upper limit of the population (1)

Given also that the population reaches 2500 at 3 years from the start of the study,

c. Calculate the value of k to 3 decimal places. (5)

Using this value for k ,

d. Find the population at 10 years from the start of the study, giving your answer to 3 significant figures (2)

24. Find algebraically the exact solutions to the equations

a. $\ln(4 - 2x) + \ln(9 - 3c) = 2\ln(x + 1) \quad -1 < x < 2$ (5)

b. $2^x e^{3x+1} = 10$

Give your answer to (b) in the form $\frac{a+\ln b}{c+\ln d}$ where $a, b, c,$ and d are integers (5)



Mark Scheme

1.

$\log_{10}1000 = 3$	M1
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2.

$\log_a(4 \times 7) = \log_a28$	M1
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3.

$\log_q x^{-1/2}$ $= -2\log_q x$	M1
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4.

$= \log_q 8^2$ $= 2y$	M1
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5.

$\frac{1}{2}x + 1 = 4^{3.2}$	M1
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$x = 2(4^{3.2} - 1)$ $x = 167$	M1
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6.

$\log_3 5x = \log_3(2x + 3)$	M1
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$5x = 2x + 3$ $x = 1$	M1
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7.

$\log_y 32 = -\frac{5}{2}$ $y^{-2.5} = 32$	M1
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$y = 32^{-0.4} = \frac{1}{4}$	M1
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$\log_2 x = 3 - 2\log_2 \frac{1}{4}$	M1
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$\log_2 x = 3 - (-4) = 7$ $x = 2^7 = 128$	M1
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$x = 128$ $y = \frac{1}{4}$	M1
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8.

$x = \log 14 = 1.15$	M1
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9.

$(4 + x)\log 3 = \log 16$	M1
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$x = \frac{\log 16}{\log 3} - 4$	M1
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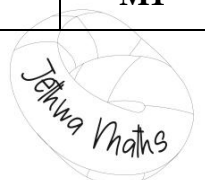
$x = -1.48$	M1
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10.

$3^{x+1} = 2^{4+x}$	M1
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$(x + 1)\log 3 = (4 + x)\log 2$	M1
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$x(\log 3 - \log 2) = 4 \log 2 - \log 3$ $x = 4.13$	M1
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11.

$3(3^{2x}) - 18(3^x) + 15 = 0$ $3(3^x - 1)(3^x - 5) = 0$	M1
$3^x = 1$ $x = 0$	M1
$3^x = 5$	M1
$x = \frac{\log 5}{\log 3}$	M1

12.

Shape of $y = 2^{x+3}$ M1 Shape of $y = 2^x$ M1 Point (0,8) and (0,1) labelled. M1	
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13.

$= \log_{10} \frac{3}{2}$	M1
$= \log_{10} 3 - \log_{10} 2$ $= b - a$	M1

14a.

$2\log_2(x + 15) - \log_2 x = 6$ $\log_2(x + 15)^2 - \log_2 x = 6$	M1
$\log_2 \frac{(x+15)^2}{x} = 6$ $\frac{(x+15)^2}{x} = 2^6$	M1
$(x + 15)^2 = 64x$ $x^2 + 30x + 225 = 64x$	M1
$x^2 - 34x + 225 = 0$	M1

14b.

$x^2 - 34x + 225 = 0$ $(x - 25)(x - 9) = 0$	M1
$x = 25$ $x = 9$	M1

15a.

$\log 5^x = \log 7$ $x \log 5 = \log 7$	M1
$x = \frac{\log 7}{\log 5}$ $x = 1.21$ (to 3 s.f)	M1

15b.

$5^{2x} - 12(5^x) + 35 = 0$ $(5^x)^2 - 12(5^x) + 35 = 0$	M1
let $y = 5^x$ $y^2 - 12y + 35 = 0$ $(y - 5)(y - 7) = 0$	M1
$y = 5$ $5^x = 5$ $x = 1$	M1
$y = 7$ $5^x = 7$ $x = 1.21$ (to 3 s.f)	M1

16a.

When $t = 0, m = 7.5$ Therefore, $7.5 = pe^0$ $p = 7.5$	M1
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16b.

When $t = 4, m = 2.5$ $2.5 = 7.5e^{-4k}$	M1
$e^{-4k} = \frac{2.5}{7.5} = \frac{1}{3}$	M1
$\ln\left(\frac{1}{3}\right) = \ln e^{-4k}$ $\ln\left(\frac{1}{3}\right) = -4k \ln e$ $\ln 1 - \ln 3 = -4k$ $-\ln 3 = -4k$	M1
$k = \frac{1}{4} \ln 3$	M1

17a.

425°C	M1
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17b.

$300 = 400e^{-0.05t} + 25$	M1
400	M1

18a.

$t = 0$ $T(0) = T_R + (90 - T_R)e^{-\frac{1}{20} \times 0}$ $= T_R + (90 - T_R)e^0$ $= T_R + 90 - T_R$ $= 90$	M1
The room temperature T_R will always cancel itself out in $T(0)$, so the initial temperature is independent of T_R .	M1

18b.

Substitute $T_R = 20$ and $t = 10$ into $T(t)$ $T(t = 10, T_R = 20) = 20 + (90 - 20)e^{-\frac{1}{20} \times 10}$	M1
$= 20 + 70e^{-\frac{1}{2}}$ $= 60.457^\circ\text{C}$	M1

19.

$\log_3(3b - 1) - \log_3(a - 2) = -1$ $\log_3 \frac{3b-1}{a-2} = -1$	M1
$\frac{3b-1}{a-2} = 3^{-1}$ $3b - 1 = \frac{1}{3}(a - 2)$	M1
$b = \frac{1}{9}a - \frac{5}{9}$	M1

20a.

$\theta = 20 + Ae^{-kt}$ $t = 0, \theta = 90$ $90 = 20 + Ae^{-kt}$	M1
$90 = 20 + A$ $A = 70$	M1

20b.

$\theta = 20 + 70e^{-kt}$ $t = 5, \theta = 55$ $\frac{35}{70} = e^{-kt}$	M1
$\ln\left(\frac{35}{70}\right) = -5k$	M1
$-5k = \ln \frac{1}{2}$ $-5k = \ln 1 - \ln 2$ $-5k = -\ln 2$	M1
$k = \frac{1}{5} \ln 2$	M1

21a.

20 mm ²	M1
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21b.

$40 = 20 e^{1.5t}$ $e^{1.5t} = c$ $e^{1.5t} = \frac{40}{20}$	M1
$1.5t = \ln 2$ $t = \frac{\ln c}{1.5}$	M1
$t = \frac{\ln 2}{1.5} = 0.486$	M1
therefore $t = 12.28$ or 28 minutes.	M1

22a.

£19500	M1
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22b.

$9500 = 17000e^{-0.25t} + 2000e^{-0.5t} + 500$	M1
$17e^{-0.25t} + 2e^{-0.5t} = 9$	
$17e^{0.25t} + 2 = 9e^{0.5t}$	M1
$0 = 9e^{0.5t} - 17e^{0.25t} - 2$	M1
$0 = (9e^{0.25t} + 1)(e^{0.25t} - 2)$	M1
$e^{0.25t} = 2$	M1
$t = 4 \ln(2)$	

23a.

$t = 0$	M1
$P = \frac{8000}{1+7} = 1000$	

23b.

$t \rightarrow \infty$	M1
$P = \frac{8000}{1} = 8000$	

23c.

$t \rightarrow 3$ $P = 2500$	M1
$2500 = \frac{8000}{1+7e^{-3k}}$	M1
$e^{-3k} = \frac{2.2}{7}$	M1
$k = -\frac{1}{3} \ln\left(\frac{2.2}{7}\right)$	M1
$k = 0.386$	M1

23d.

$t = 10$	M1
$P = \frac{8000}{(1+7e^{-0.386t})}$	
$P = 6970$	M1

24a.

$\ln(4 - 2x)(9 - 3x) = \ln(x + 1)^2$	M1 M1
$36 - 30x + 6x^2 = x^2 + 2x + 1$	M1
$5x^2 - 32x + 35 = 0$	
$x = 5 \rightarrow$ solution not valid.	M1
$x = \frac{7}{5}$	M1

24b.

$\ln 2^x + \ln e^{3x+1} = \ln 10$	M1
$x \ln 2 + (3x + 1) \ln e = \ln 10$	M1
$x(\ln 2 + 3 \ln e) = \ln 10 - \ln e$	M1
$\ln e = 1$	M1
$x = \frac{-1 + \ln 10}{3 + \ln 2}$	M1

