AS-Level Unit Test: Differentiation

1. Differentiate $12x^3 + 4x^{-1}$ with respect to x **(2)**

2. Find f'(x) when f(x) = $2x^{\frac{1}{6}} + x^{\frac{3}{4}}$ **(2)**

3. f(x) = (x + 1)(x + 6). Find the derivative of the function **(2)**

4. Find $\frac{dy}{dx}$ when $y = \frac{8x+x^3}{4\sqrt{x}}$ **(3)**

5. Find the gradient of the curve $y = 3x^2 + x - 5$ at the points (2, 9) **(2)**

6. A curve has the equation $y = x^2 - 3x + 4$.
a. Find an equation of the normal to the curve at the point A (2, 2). The normal to the curve at A intersects the curve again at the point B. **(3)**
b. Find the coordinates of the point B. **(4)**

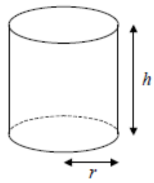
7. The line with equation $y = 2x + k$ is a normal to the curve with the equation $y = \frac{16}{x^2}$. Find the value of the constant k. **(5)**

8. A ball is thrown vertically downwards from the top of a cliff. The distance, s metres, of the ball from the top of the cliff after t seconds is given by $s = 3t + 5t^2$. Find the rate at which the distance the ball has travelled is increasing when
a. $t = 0.6$ **(2)**
b. $s = 54$ **(4)**

9. Given that $y = \frac{x^2-6x-3}{3x^{\frac{1}{2}}}$, show that $\frac{dy}{dx}$ can be expressed in the form $\frac{(x+a)^2}{bx^{\frac{1}{2}}}$ where a and b are integers to be found. **(6)**

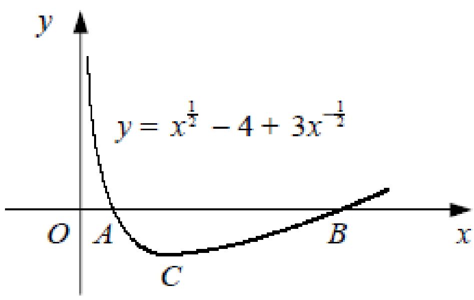
10. Find the coordinates of the stationary point of the curve $y = x^3 + 3x^2 + 3x$ and determine its nature. **(7)**

11. The diagram shows a closed plastic cylinder used for making compost. The radius of the base and the height of the cylinder are r cm and h cm respectively and the surface area of the cylinder is 30 000 cm^2.



a. Show that the volume of the cylinder, V cm^3, is given by $V = 15\,000r - \pi r^3$ **(4)**
b. Find the maximum volume of the cylinder and show that your value is a maximum **(6)**

12. The diagram shows the curve with equation $y = x^{\frac{1}{2}} - 4 + 3x^{-\frac{1}{2}}$.
The curve crosses the x-axis at
the points A and B and has a minimum point at C.
a. Find the coordinates of A and B **(3)**
b. Find the coordinates of C, giving its y-coordinate in the form $a\sqrt{3} + b$, where a and b are integers. **(4)**



13. The curve with equation $y = x^3 + ax^2 - 24x + b$, where a and b are constants, passes through the point P $(-2, 30)$.

a. Show that $4a + b + 10 = 0$. **(2)**

Given also that P is a stationary point of the curve,

b. Find the values of a and b, **(4)**

c. Find the coordinates of the other stationary point on the curve. **(3)**

14. $f(x) = 4x^3 + ax^2 - 12x + b$.

Given that a and b are constants and that when $f(x)$ is divided by $(x + 1)$ there is a remainder of 15,

a. Find the value of $(a + b)$. **(2)**

Given also that when $f(x)$ is divided by $(x - 2)$ there is a remainder of 42

b. Find the values of a and b **(5)**

c. Find the coordinates of the stationary points of the curve $y = f(x)$ **(5)**

Total marks: 45

Mark Scheme

1.

$\frac{d}{dx}(12x^3) = 36x^2$	**M1**
$\frac{d}{dx}(4x^{-1}) = -4x^{-2}$	**M1**
$\frac{d}{dx}(12x^3+4x^{-1}) = 36x^2 - 4x^{-2}$	

2.

$\frac{d}{dx}(2x^{\frac{1}{6}}) = \frac{1}{3}x^{-\frac{5}{6}}$	**M1**
$\frac{d}{dx}(x^{\frac{3}{4}}) = \frac{3}{4}x^{-\frac{1}{4}}$	**M1**
$\frac{d}{dx}(2x^{\frac{1}{6}} + x^{\frac{3}{4}}) = \frac{1}{3}x^{-\frac{5}{6}} + \frac{3}{4}x^{-\frac{1}{4}}$	

3.

$f(x) = (x + 1)(x + 6) = x^2 + 7x + 6$	**M1**
$f'(x) = 2x + 7$	**M1**

4.

$y = \frac{8x+x^3}{4\sqrt{x}} = \frac{8x+x^3}{4x^{0.5}} = \frac{8x}{4x^{0.5}} + \frac{x^3}{4x^{0.5}}$	**M1**
$y = 2x^{0.5} + \frac{1}{4}x^{2.5}$	**M1**
$\frac{dy}{dx} = x^{-0.5} + \frac{5}{8}x^{1.5}$	**M1**

5.

$y = 3x^2 + x - 5$ $\frac{dy}{dx} = 6x + 1$	**M1**
$\frac{dy}{dx}(2) = 6(2) + 1 = 13$	**M1**

6a.

$\frac{dy}{dx} = 2x - 3$ Gradient = 1	**M1**
Gradient of normal = -1	**M1**
$y - 2 = -x(x - 1)$ $y = 4 - x$	**M1**

6b.

$x^2 - 3x + 4 = 4 - x$ $x^2 - 2x = 0$	**M1**
$x(x - 2) = 0$	**M1**
$x = 2$ $x = 0$	**M1**
Therefore B = (0, 4)	**M1**

7.

Gradient of normal = 2 Therefore, gradient of curve = $-\frac{1}{2}$	**M1**

For the curve $\frac{dy}{dx} = -32x^{-3}$	**M1**
$-\frac{32}{x^{-3}} = -\frac{1}{2}$	**M1**
$x^3 = 64$ $x = 4$ Therefore, coordinate is (4, 1)	**M1**
$1 = 8 + k$ $k = -7$	**M1**

8a.

$a = \frac{ds}{dt} = 3 + 10t$ $t = 0.6$	**M1**
$\frac{ds}{dt}$ = 9 metres per second	**M1**

8b.

$54 = 3t + 5t^2$	**M1**
$5t^2 + 3t - 54 = 0$ $(5t + 18)(t - 3) = 0$	**M1**
$t = 3$ $t = -\frac{18}{5} \rightarrow$ not valid as $t > 0$	**M1**
$\frac{ds}{dt} = 33$ metres per second.	**M1**

9.

$y = \frac{1}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$	**M1** **M1**
$= \frac{x^2-2x+1}{2x^{\frac{3}{2}}}$	**M1** **M1**
$= \frac{(x-1)^2}{2x^{\frac{3}{2}}}$	**M1** **M1**
$a = -1$ $b = 2$	

10.

$\frac{dy}{dx} = 3x^2 + 6x + 3$	**M1**
Stationary point: $\frac{dy}{dx} = 0$ $3x^2 + 6x + 3 = 0$	**M1** **M1**
$3(x + 1)^2 = 0$ $x = -1$	**M1**
$\frac{d^2y}{dx^2} = 6x + 6$	**M1**
At (-1, -1), $\frac{d^2y}{dx^2} = 0$	**M1**
As $\frac{d^2y}{dx^2} = 0$, stationary point is a point of infection.	**M1**

11a.

$S.A = 2\pi r^2 + 2\pi rh = 30000$	**M1**

$\pi rh = 15000 - \pi r^2$ $h = \frac{15000}{\pi r} - r$	**M1**
$V = \pi r^2 h$ $V = \pi r^2(\frac{15000}{\pi r} - r)$	**M1**
$V = 15000r - \pi r^3$	**M1**

11b.

$\frac{dV}{dr} = 15000 - 3\pi r^2$	**M1**
Stationary point = $15000 - 3\pi r^2 = 0$	**M1**
$r^2 = \frac{5000}{\pi}$ $r = 39.9$	**M1**
Max volume = 399 000 cm^3 $\frac{d^2V}{dr^2} = -6\pi r$	**M1**
$r = 39.9$ $\frac{d^2V}{dr^2} = -752$	**M1**
As $\frac{d^2V}{dr^2} < 0$, point is a maximum	**M1**

12a.

$x^{\frac{1}{2}} - 4 + 3x^{-\frac{1}{2}} = 0$	**M1**
$x - 4x^{\frac{1}{2}} + 3 = 0$ $x^{\frac{1}{2}} = 1$ $x = 3$	**M1**
$x^{\frac{1}{2}} = 3$ $x = 9$	**M1**
Therefore: (1, 0) and (9, 0)	

12b.

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$	**M1**
$\frac{dy}{dx} = 0$ $\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}} = 0$	**M1**
$\frac{1}{2}x^{-\frac{3}{2}}(x - 3) = 0$ $x = 3$	**M1**
$y(3) = (3)^{\frac{1}{2}} - 4 + 3(3)^{-\frac{1}{2}} = 2\sqrt{3} - 4$	**M1**
$(3, 2\sqrt{3} - 4)$	

13a.

At (-2, 30) → $30 = -8 + 4a + 48 + b$	**M1**

Therefore, $4a + b + 10 = 0$	**M1**

13b.

$\frac{dy}{dx} = 3x^2 + 2ax - 24$	**M1**
As P is a stationary point, $\frac{dy}{dx} = 0$	**M1**
$12 – 4a – 24 = 0$ $a = -3$ $b = 2$	**M1** **M1**

13c.

$3x^2 – 6x – 24 = 0$ $3(x + 2)(x – 4) = 0$	**M1**
$x = -2$ (already given) $x = 4$	**M1**
At $x = 4$, $y = -78$	**M1**
Other stationary point is (4, -78)	

14a.

f(-1) = 15 $-4 + a + 12 + b = 15$	**M1**
$a + b = 7$	**M1**

14b.

f(2) = 42 $32 + 4a – 24 + b = 42$	**M1**
$4a + b = 34$	**M1**
Solving simultaneously, $3a = 27$ $a = 9$	**M1** **M1**
$b = -2$	**M1**

14c.

$f(x) = 4x^3 + 9x^2 – 12x – 2$	**M1**
$f'(x) = 12x^2 + 18x – 12$ Stationary point: $f'(x) = 0$ $12x^2 + 18x – 12 = 0$	**M1**
$(2x – 1)(x + 2) = 0$ $x = -2$ $x = \frac{1}{2}$	**M1**
at $x = -2$, $y = 26$	**M1**
at $x = \frac{1}{2}, y = -\frac{21}{4}$	**M1**
(-2, 26) and $(\frac{1}{2}, -\frac{21}{4})$	