Differentiation

Pt. 1: First Principles

1. Prove from first principles that the derivative of x^3 is $3x^2$ **(5)**

2. Prove, from first principles, that the derivative of kx^3 is $3kx^2$. Where k is a constant. **(5)**

3. Differentiate from first principles $y = 2x^2$ **(5)**

Mark Scheme

1.

$y = x^3$ $x_1 = x$ $y_1 = x^3$ $x_2 = x + h$ $y_2 = (x + h)^3$	**M1**
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(x+h)^3 - x^3}{x+h-x}$	**M1**
$m = \frac{(x+h)(x+h)(x+h) - x^3}{h}$ $m = \frac{x^3+hx^2+2hx^2+2h^2x+h^2x+h^3 - x^3}{h}$	**M1**
$m = \frac{3hx^2+3hx^2+h^3}{h}$ $m = 3x^2 + 3hx + h^2$	**M1**
As h tends to 0, $m = 3x^2$	**M1**

2.

$y = kx^3$ $x_1 = x$ $y_1 = kx^3$ $x_2 = x + h$ $y_2 = k(x + h)^3$	**M1**
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k(x+h)^3 - kx^3}{x+h-x}$	**M1**
$m = \frac{k(x+h)(x+h)(x+h) - kx^3}{h}$ $m = \frac{kx^3+hkx^2+2hkx^2+2h^2kx+h^2kx+kh^3 - kx^3}{h}$	**M1**
$m = \frac{3hkx^2+3hkx^2+kh^3}{h}$ $m = 3kx^2 + 3hkx + kh^2$	**M1**
As h tends to 0, $m = 3kx^2$	**M1**

3.

$y = 2x^2$ $x_1 = x$ $y_1 = 2x^2$ $x_2 = x + h$ $y_2 = 2(x + h)^2$	**M1**
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2(x+h)^2 - 2x^2}{x+h-x}$	**M1**
$m = \frac{2(x+h)(x+h) - 2x^2}{h}$ $m = \frac{2x^2+4hx+2h^2 - 2x^2}{h}$	**M1**
$m = \frac{4hx+2h^2}{h}$ $m = 4x + 2h^2$	**M1**
As h tends to 0, $m = 4x$	**M1**

Differentiation

Pt. 2: Differentiation

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1. Find $\frac{dy}{dx}$ when $y = 6x^3 + 5x^{-2}$ **(2)**

2. Find f'(x) when f(x) = $2x + \frac{1}{3}x^6$ **(2)**

3. Find the differential of $y = \frac{x^3 - 2x}{x}$ **(2)**

4. Find the differential of $y = \frac{x+3}{\sqrt{x}}$ **(3)**

5. Find the gradient of the curve, $y = x^2 - 2x^{-1}$ at the point (2, 3) **(3)**

6. Find the equation of the tangent to the curve $y = x^2 + 3x + 4$ at the point (-1, 2) **(3)**

7. Find in the form $y = mx + c$, an equation of,
a. The tangent to the curve $y = 3x^2 - 5x + 2$ at the point on the curve with x-coordinate 2. **(4)**
b. The normal to the curve $y = x^3 + 5x^2 - 12$ at the point on the curve with x-coordinate -3. **(4)**

8. The straight line l is a tangent to the curve $y = x^2 - 5x + 3$ at the point A on the curve.
Given that l is parallel to the line $3x + y = 0$,
a. Find the coordinates of the point A **(3)**
b. Find the equation of the line l in the form $y = mx + c$. **(2)**

9. The curve C has the equation $y = x - 3x^{\frac{1}{2}} + 3$ and passes through the point P (4, 1).
a. Show that the tangent to C at P passes through the origin. **(4)**
The normal to C at P crosses the y-axis at the point Q.
b. Find the area of triangle OPQ, where O is the origin. **(4)**

10. A curve has the equation $y = 2 + 3x + kx^2 - x^3$ where k is a constant.
Given that the gradient of the curve is −6 at the point P where $x = -1$,
a. Find the value of k. **(4)**
Given also that the tangent to the curve at the point Q is parallel to the tangent at P
b. Find the length PQ, giving your answer in the form $k\sqrt{5}$. **(5)**

11. Find the set of value of x for which f(x) is increasing when f(x) = $x^3 + 6x^2 - 15x + 8$ **(3)**

12. Find the co-ordinates of any stationary points on the curve $y = 5x^2 - 4x + 1$ **(4)**

13. Find the stationary point of the curve $y = x^2 + \frac{16}{x}$ and states its nature. **(6)**

14. Sketch $y = 3x - 4x^{\frac{1}{2}}$ **(7)**

15. The diagram shows a square prism of length l cm and cross section x cm by x cm. Given that the surface area of the prism is k cm^2, where k is a constant,

a. Show that l $= \frac{k-2x^2}{4x}$ **(2)**

b. Use calculus to prove that the maximum volume of the prism occurs when it is a cube **(7)**

16. The curve C has the equation $y = x^3 + 3kx^2 - 9k^2x$, where k is a non-zero constant.

a. Show that C is stationary when, $x^2 + 2kx - 3k^2 = 0$. **(2)**

b. Hence show that C is stationary at the point with cooridinates $(k, -5k^3)$. **(2)**

c. Find, in terms of k, the coordinates of the other stationary points on C. **(2)**

17. $f(x) = x^3 - 3x^2 + 4$

a. Show that $(x + 1)$ is a factor of $f(x)$. **(1)**

b. Fully factorise $f(x)$. **(2)**

c. Hence state, with a reason, the coordinates of one of the turning points of the curve $y = f(x)$. **(1)**

d. Using differentiation, find the coordinates of the other turning point of the curve $y = f(x)$. **(3)**

18. The curve C has the equation $y = 2x - x^{\frac{3}{2}}$, $x \geq 0$

a. Find the cooridnates of any points where C meets the x-axis. **(3)**

b. Find the x-cordinate of the stationary point on C and determine whether it is a maximum or minimum point **(5)**

c. Sketch the curve C **(2)**

19a. Find the coordinates of the stationary points on the curve, $y = 2 + 9x + 3x^2 - x^3$. **(6)**

b. Determine whether each stationary point is a maximum or minimum point. **(2)**

c. State the set of values of k for which the equation, $2 + 9x + 3x^2 - x^3 = k$, has three solutions. **(2)**

Mark Scheme

1.

$\frac{dy}{dx} = 18x^2 - 10x^{-3}$	**M1 M1**

2.

$f'(x) = 2 + 2x^5$	**M1 M1**

3.

$y = \frac{x^3}{x} - \frac{2x}{x} = x^2 - 2$	**M1**
$\frac{dy}{dx} = 2x$	**M1**

4.

$y = \frac{x}{\sqrt{x}} - \frac{3}{\sqrt{x}}$	**M1**
$y = \frac{x}{x^{0.5}} - \frac{3}{x^{0.5}} = x^{0.5} - 3x^{-0.5}$	**M1**
$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{5}{2}}$	**M1**

5.

$\frac{dy}{dx} = 2x + 2x^{-2}$	**M1**
At $x = 2$, $\frac{dy}{dx} = 2(2) + 2(2)^{-2}$	**M1**
$\frac{dy}{dx} = 12$	**M1**

6.

$\frac{dy}{dx} = 2x + 3$	**M1**
At $x = -1$, $\frac{dy}{dx} = 2(-1) + 3 = 1$	**M1**
$x = -1$, $y = 2$, $m = 1$ $2 = (1)(-1) + c$ $2 = -1 + c$ $c = 3$	**M1**
$y = x + 3$	

7a.

$\frac{dy}{dx} = 6x - 5$	**M1**
At $x = 2$, $\frac{dy}{dx} = 6(2) - 5 = 7$	**M1**
At $x = 2$, $y = 3(2)^2 - 5(2) + 2 = 4$	**M1**
$y = 4$, $m = 7$, $x = 2$ $4 = (7)(2) + c$ $c = -10$	**M1**

$y = 7x - 10$	

7b.

$\frac{dy}{dx} = 3x^2 + 10x$	**M1**
At $x = -3$, $\frac{dy}{dx} = 3(-3)^2 + 10(-3) = -3$ Gradeint of normal = $\frac{1}{3}$	**M1**
At $x = -3$, $y = (-3)^3 + 5(-3)^2 - 12 = 6$	**M1**
$y = 6$, $m = \frac{1}{3}$, $x = -3$ $6 = (\frac{1}{3})(-3) + 6$ $c = 7$	**M1**
$y = \frac{1}{3}x + 7$	

8a.

Gradient of line = -3	**M1**
For the curve, $\frac{dy}{dx} = 2x - 5$	**M1**
At the point A, $2x - 5 = -2$ $x = -1$	**M1**
Therefore *A:* (1, -1)	

8b.

$y + 1 = -3(x - 1)$	**M1**
$y = -3x + 2$	**M1**

9a.

$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}}$	**M1**
Gradient at $P = \frac{1}{4}$	**M1**
$y - 1 = \frac{1}{4}(x - 4)$	**M1**
As line passes through origin $y = \frac{1}{4}x$	**M1**

9b.

Gradient of normal = -4	**M1**
$y - 1 = -4(x - 4)$ $y = 17 - 4x$	**M1**
$y(0) = 17 - 4(0) = 17$	**M1**
Therefore area = ½ x 17 x 4 = 34	**M1**

10a.

$\frac{dy}{dx} = 3 + 2kx - 3x^2$	**M1 M1**
At P, $3 - 2k - 3 = -6$	**M1**
$k = 3$	**M1**

10b.

$y = 2 + 3x + 3x^2 - x^3$	**M1**
At point P, $x = -1$, $y = 3$	**M1**

10b.

10c.

At Q, $3 + 6x - 3x^2 = -6$ $x^2 - 2x - 3 = 0$ $(x + 1)(x - 3) = 0$ $x = -1$ (at P) $x = 3$ (at Q) $y = 11$	**M1**
$PQ = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}$	**M1**

11.

$f'(x) = 3x^2 + 12x - 15$ $3x^2 + 12x - 15 \geq 0$	**M1**
$x^2 + 4x - 5 \geq 0$ $(x + 5)(x - 1) \geq 0$	**M1**
$x \leq -5$ $x \geq 1$	**M1**

12.

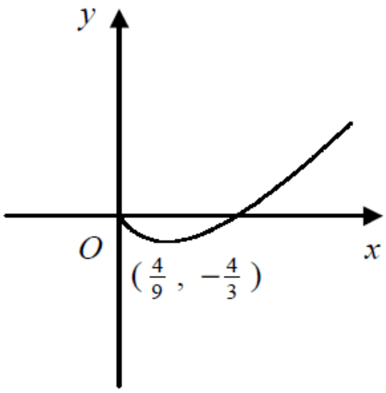
$\frac{dy}{dx} = 10x - 4$	**M1**
At stationary point, $\frac{dy}{dx} = 0$	**M1**
$10x - 4 = 0$ $x = \frac{2}{5}$	**M1**
Therefore stationary point is $(\frac{2}{5}, \frac{1}{5})$	**M1**

13.

$\frac{dy}{dx} = 2x - 16x^{-2}$	**M1**
At stationary point, $\frac{dy}{dx} = 0$	**M1**
$2x - 16x^{-2} = 0$ $x^3 = 8$ $x = 2$	**M1**
$\frac{d^2y}{dx^2} = 2 + 32x^{-3}$	**M1**
$\frac{d^2y}{dx^2}(2) = 2 + 32(2)^{-3} = 6$	**M1**

As $\frac{d^2y}{dx^2} > 0$, stationary point is a minimum	**M1**

14.

$\frac{dy}{dx} = 3 - 2x^{-\frac{1}{2}}$	**M1**
At stationary point, $\frac{dy}{dx} = 0$	**M1**
$3 - 2x^{-\frac{1}{2}} = 0$ $x = \frac{4}{9}$	**M1**
$\frac{d^2y}{dx^2} = x^{-\frac{3}{2}}$	**M1**
$\frac{d^2y}{dx^2}\left(\frac{4}{9}\right) = \left(\frac{4}{9}\right)^{-\frac{3}{2}} = \frac{27}{8}$	**M1**
As $\frac{d^2y}{dx^2} > 0$, stationary point is a minimum	**M1**
 Shape **M1**	

15a.

S.A = $2x^2 + 4x$l = k $4x$l = k – $2x^2$	**M1**
l = $\frac{k-2x^2}{4x}$	**M1**

15b.

V = x^2l V = x^2 x $\frac{k-2x^2}{4x}$ V = $\frac{1}{4}kx - \frac{1}{2}x^3$	**M1**
$\frac{dV}{dx} = \frac{1}{4}k - \frac{3}{2}x^2$	**M1**
At stationary point, $\frac{dV}{dx} = 0$ $\frac{1}{4}k - \frac{3}{2}x^2 = 0$ $x = \sqrt{\frac{k}{6}}$	**M1**
$\frac{d^2V}{dx^2} = -3x$	**M1**
When $x = \sqrt{\frac{k}{6}}$, $\frac{d^2V}{dx^2}$ = < 0, therefore stationary point is a maximum	**M1**
l = $\frac{k-2\left(\sqrt{\frac{k}{6}}\right)^2}{4\sqrt{\frac{k}{6}}}$	**M1**

$l = \frac{2}{3}k \times \frac{1}{4} \times \sqrt{\frac{6}{k}}$ $= \frac{k}{6} \times \sqrt{\frac{6}{k}} = \sqrt{\frac{k}{6}}$	
Therefore maximum V occurs when l = x, therefore prism is a cube	**M1**

16a.

$\frac{dy}{dx} = 3x^2 + 6kx - 9k^2$	**M1**
At stationary point, $\frac{dy}{dx} = 0$ $3x^2 + 6kx - 9k^2 = 0$ $x^2 + 2kx - 3k^2 = 0$	**M1**

16b.

$(x + 3k)(x - 3k) = 0$ $x = -3k$ $x = k$	**M1**
When $x = k$, $y = k^3 + 3k^3 - 9k^3 = -5k^3$ Therefore station point is at $(k, -5k^3)$	**M1**

16c.

When $x = -3k$ $y = -27k^3 + 27k^3 + 27k^3$	**M1**
Therefore point is $(-3k, 27k^3)$	**M1**

17a.

f(-1) = -1 – 3 + 4 = 0 Therefore $(x + 1)$ is a factor	**M1**

17b.

$\begin{array}{r l} & x^2 - 4x + 4 \\ x+1 \,) & x^3 - 3x^2 + 0x + 4 \\ & x^3 + x^2 \\ & -4x^2 + 0x \\ & -4x^2 - 4x \\ & 4x + 4 \\ & 4x + 4 \end{array}$	**M1**
$f(x) = (x + 1)(x^2 - 4x + 4)$ $f(x) = (x + 1)(x - 2)^2$	**M1**

17c.

(2, 0) as $(x - 2)$ is a repeated factor or $f(x)$ so x-axis is a tangent at (2, 0)	**M1**

17d.

$f'(x) = 3x^2 - 6x$	**M1**
Stationary point = $3x^2 - 6x = 0$ $3x(x - 2) = 0$ $x = 0$ $x = 2$	**M1**
When $x = 0$, $y = 4$ Therefore, other turning point is (0, 4)	**M1**

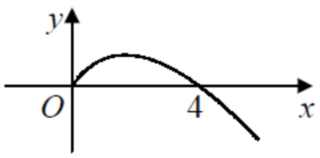
18a.

$2x - x^{\frac{3}{2}} = 0$ $x(2 - x^{\frac{1}{2}}) = 0$	**M1**
$x = 0$ $y = 0$	**M1**
$2 - x^{\frac{1}{2}}$ $x = 4$ $y = 0$	**M1**
(0, 0) and (4, 0)	

18b.

$\frac{dy}{dx} = 2 - \frac{3}{2}x^{\frac{1}{2}}$	**M1**
Stationary point: $2 - \frac{3}{2}x^{\frac{1}{2}} = 0$ $x^{\frac{1}{2}} = \frac{4}{3}$ $x = \frac{16}{9}$	**M1**
$\frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{1}{2}}$	**M1**
when $x = \frac{16}{9}$ $\frac{d^2y}{dx^2} = -\frac{9}{16}$	**M1**
$\frac{d^2y}{dx^2} < 0$, therefore TP is a minimum	**M1**

18c.

Shape **M1** (4, 0) labelled **M1** 	

19a.

$\frac{dy}{dx} = 9 + 6x - 3x^2$	**M1** **M1**
Stationary point: $9 + 6x - 3x^2 = 0$ $-3(x + 1)(x - 3) = 0$	**M1**
$x = -1$, y -3	**M1 M1**
$x = 3$, y 29	**M1**
(-1, -3) (3, 29)	

19b.

$\frac{d^2y}{dx^2} = 6 - 6x$	**M1**

at (-1, -3): $\frac{d^2y}{dx^2}$ = 12, therefore a minimum point	**M1**
at (3, 29): $\frac{d^2y}{dx^2}$ = -12, therefore a maximum point	**M1**

19c.

$-3 < k < 29$	**M1**