Sequences and Series

Part 1b: Binomial Expansion

AS Level	A-Level	
Part 1a: Binomial Expansion 1	Part 1b: Binomial Expansion 2	Part 3: Geometric Sequences
	Part 2: Arithmetic Sequences	Part 4: Recurrence Relations

1a. Expand $(1 + 2x)^{0.5}$, $|x| < 0.5$, in ascending powers of x up to and including the term x^3 **(4)**

b. By substituting a suitable value of x in the expansion, find an estimate for $\sqrt{0.98}$ **(2)**

c. Show that $\sqrt{0.98} = \frac{7}{10}\sqrt{2}$ and hence find an estimate for $\sqrt{2}$ correct to 8 significant figures. **(2)**

2. Expand $(1 - 3x)^{-\frac{4}{3}}$, in ascending powers of x up to and including the term x^3 **(3)**

b. State the set of values of x for which the expansion is valid. **(2)**

3a. Expand $(4 + 6x)^{-\frac{1}{2}}$, in ascending powers of x up to and including the term x^3 **(4)**

b. State the set of values of x for which the expansion is valid. **(2)**

4. Find the first four terms of the series expansion $\frac{2x-1}{(1+4x)^2}$ and state the set of values of x for which each expansion is valid.

(5)

5. Find the first four terms of the series expansion $\frac{3+x}{2-x}$ and state the set of values of x for which each expansion is valid.

(6)

6a. Express $\frac{x-2}{(1-x)(1-2x)}$ in partial fractions. **(3)**

b. Hence, find the series expansion of $\frac{x-2}{(1-x)(1-2x)}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid.

(8)

7. Find the series expansion of $f(x) = \frac{2x^2+4}{2x^2+x-1}$ and state the set of values for which it is valid. **(10)**

8a. Find the binomial expansion of $(4 + x)^{0.5}$ in ascending powers of x up to and including the term in x^2 and state the set of values of x for which the expansion is valid.

(4)

b. By substituting $x = \frac{1}{20}$ in your expansion, find an estimate for $\sqrt{5}$, giving your answer to 9 significant figures.

(3)

c. Obtain the value of $\sqrt{5}$ from your calculator and hence comment on the accuracy of the estimate found in part b.

(2)

<u>Mark Scheme</u>

1a.

$(1 + 2x)^{.5} = 1 + (\frac{1}{2})(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-2x)^3 + \ldots.$	**M1 M1**
$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \ldots$	**M1 M1**

1b.

$\sqrt{0.98} = (1 - 2x)^{0.5}$ when $x = 0.01$	**M1**
$\sqrt{0.98} \approx 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$	**M1**
$= 1 - 0.01 - 0.00005 - 0.0000005$ $= 0.9899495$	**M1**

1c.

$\sqrt{0.98} \approx \sqrt{\frac{98}{100}} = \sqrt{\frac{49 \times 2}{100}} = \frac{7}{10}\sqrt{2}$ $\sqrt{2} \approx \frac{10}{7} \times 0.9899495 = 1.4142136$ (8 s.f)	**M1**

2a.

$(1 - 3x)^{-\frac{4}{3}} = 1 + -\frac{4}{3}(-3x) + \frac{(-\frac{4}{3})(-\frac{7}{3})}{2}(-3x)^2 + \frac{(-\frac{4}{3})(-\frac{7}{3})(-\frac{10}{3})}{3 \times 2}(-3x)^3 + \ldots.$	**M1**
$1 + 4x + 14x^2 + \frac{140}{3}x^3$	**M1 M1**

2b.

$\lvert -3x \rvert < 1$,	**M1**
Therefore expansion is valid for $\lvert x \rvert < \frac{1}{3}$	**M1**

3a.

$(4 + 6x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}(1 + \frac{3}{2}x)^{-0.5} = \frac{1}{2}(1 + \frac{3}{2}x)^{-0.5}$	**M1**
$= \frac{1}{2}[1 + \left(-\frac{1}{2}\right)\left(\frac{3}{2}x\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{3}{2}x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3 \times 2}(\frac{3}{2}x)^3 + \ldots.]$	**M1**
$= ½ - \frac{3}{8}x + 2764x^2 - \frac{32}{243}x^3 + \ldots..$	**M1 M1**

3b.

$(4 + 6x)$ valid for $\left\lvert \frac{3}{2}x \right\rvert < 1$	**M1**
Expansion is valid for $\lvert x \rvert < \frac{3}{2}$	**M1**

4.

$\frac{2x-1}{(1+4x)^2} = (2x - 1)(1 + 4x)^{-2}$	**M1**
$=(2x - 1)(1 + (-2)(4x) + \frac{(-2)(-3)}{2}(4x)^2 + \frac{(-2)(-3)(-4)}{3\times2}(4x)^3$	**M1**
$= (2x - 1)(1 - 8x + 48x^2 - 256x^3)$	**M1**
$= 2x - 16x^2 + 96x^3 - 1 + 8x - 48x^2 + 256x^3 + \ldots.$ $= -1 + 10x - 64x^2 + 352x^3$	**M1**
$\lvert 4x \rvert < 1$ $\lvert x \rvert < \frac{1}{4}$	**M1**

5.

$\frac{3+x}{2-x} = (3+x)(2-x)^{-1} = (3+x) \times 2^{-1}(1 - \frac{1}{2}x)^{-1}$	**M1**
$= (3+x) \times \frac{1}{2}[1 + (-1)(-\frac{1}{2}x) + \frac{(-1)(-2)}{2}(-\frac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-\frac{1}{2}x)^3 + \ldots]$	**M1**
$= (3+x)(\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots)$	**M1**
$= \frac{3}{2} + \frac{3}{4}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \ldots.$	**M1**
$= \frac{3}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3 + \ldots$	**M1**
$\left\|-\frac{1}{2}x\right\| < 1$ $\|x\| < 2$	**M1**

6a.

$\frac{x-2}{(1-x)(1-2x)} \equiv \frac{A}{1-x} + \frac{B}{1-2x}$ $x - 2 = A(1-2x) + B(1-x)$	**M1**
$x = 1$, $-1 = A(-1) + B(0)$ $A = 1$	**M1**
$x = 0.5$, $-1.5 = A(0) + B(0.5)$ $B = -3$	**M1**
$\frac{x-2}{(1-x)(1-2x)} \equiv \frac{1}{1-x} - \frac{3}{1-2x}$	

6b.

$\frac{x-2}{(1-x)(1-2x)} \equiv \frac{1}{1-x} - \frac{3}{1-2x} \equiv (1-x)^{-1} - 3(1-2x)^{-1}$	**M1**
$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-x)^3 + \ldots$	**M1**
$= 1 + x + x^2 + x^3 \ldots.$	**M1**
$\|-x\| < 1$ $\|x\| < 1$	**M1**
$3(1-2x)^{-1} = 3[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-2x)^3 + \ldots.]$ $= 3 + 6x + 12x^2 + 24x^3 + \ldots$	**M1**
$\|-2x\| < 1$ $\|x\| < \frac{1}{2}$	**M1**
$\frac{x-2}{(1-x)(1-2x)} = (1 + x + x^2 + x^3) - (3 + 6x + 12x^2 + 24x^3) = -2 - 5x - 11x^2 - 23x^3$	**M1**
As $\|x\| < \frac{1}{2}$ is more restrictive, the expansion is valid for $\|x\| < \frac{1}{2}$	**M1**

7.

Long division to find constant: $$\begin{array}{r\|l} & 1 \\ 2x^2 + x - 1 & 2x^2 + 0x + 4 \\ & 2x^2 + x - 1 \\ \hline & -x + 5 \end{array}$$	**M1**
$\frac{2x^2+4}{2x^2+x-1} \equiv 1 + \frac{5-x}{2x^2+x-1}$	**M1**
$\frac{5-x}{2x^2+x-1} = \frac{A}{2x-1} + \frac{B}{x+1}$ $5 - x = A(x+1) + B(2x-1)$	**M1**
$x = 0.5$, $4.5 = 1.5(A) + B(0)$ $A = 3$	**M1**
$x = -1$, $6 = A(0) + -3(B)$ $B = -2$	**M1**

$\frac{2x^2+4}{2x^2+x-1} \equiv 1 + \frac{3}{2x-1} - \frac{2}{x+1}$	
$\frac{3}{2x-1} = 3(1-2x)^{-1} = 3[1 + (-1)(-2x) \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(-2x)^3 + \ldots]$ $= 3 + 6x + 12x^2 + 24x^3 + \ldots$	**M1**
$\frac{3}{2x-1}$ valid for $\lvert -2x \rvert < 1$ therefore, $\lvert x \rvert < \frac{1}{2}$	**M1**
$\frac{2}{x+1} = 2(x+1)^{-1} = 2[1 + (-1)(x) \frac{(-1)(-2)}{2}(x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(x)^3 + \ldots]$ $= 2 - 2x + 2x^2 + 2x^3 + \ldots.$	**M1**
$\frac{2}{x+1}$ valid for $\lvert x \rvert < 1$	
$\frac{2x^2+4}{2x^2+x-1} = 1 - (3 + 6x + 12x^2 + 24x^3) - (2 - 2x + 2x^2 + 2x^3)$ $= -4 - 4x - 14x^2 - 22x^3 + \ldots.$	**M1**
As $\lvert x \rvert < \frac{1}{2}$ is more restrictive, the expansion is valid for $\lvert x \rvert < \frac{1}{2}$	**M1**

8a.

$4^{0.5}(1 + 0.25x)^{0.5} = 2(1 + 0.25x)^{0.5}$	**M1**
$= 2[1 + (\frac{1}{2})\left(\frac{1}{4}x\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\frac{1}{4}x)^2 + \ldots]$	**M1**
$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \ldots.$	**M1**
Valid for $\left\lvert \frac{1}{4}x \right\rvert < 1$ $\lvert x \rvert < 4$	**M1**

8b.

When x = $\frac{1}{20}$, $(4 + x)^{0.5} \approx 2 + (\frac{1}{4})(\frac{1}{20}) - (\frac{1}{64})(\frac{1}{20})^2 = 2.012460938$	**M1**
$(4 + \frac{1}{20})^{0.5} = \sqrt{\frac{81}{20}} = \sqrt{\frac{81 \times 5}{20}} = \frac{9}{10}\sqrt{5}$	**M1**
Therefore $\sqrt{5} \approx \frac{10}{9} \times 2.012460938 = 2.23606771$	**M1**

8c.

$\sqrt{5} = 2.236067977$	**M1**
Therefore, estimate is accurate to 7 significant figures	**M1**

Part 2: Arithmetic Sequences

1. The first and third terms of an arithmetic series are 21 and 27 respectively.
a. Find the common difference of the series
b. Find the sum of the first 40 terms of the series.

2. The nth term of an arithmetic series is given by $7n + 16$. Find the first term of the series and the sum of the first 35 terms of the series.

3. In an arithmetic progression the sum of the first ten terms is 400 and the sum of the next ten terms is 1000. Find the common difference and the first term. **(5)**

4. Three consecutive terms of an arithmetic series are a, b, and $(3a + 4)$ respectively. Find an expression for b in terms of a. **(2)**

5. A company, which is making 200 mobile phones each week, plans to increase its production. The number of mobile phones is to be increased by 20 each week from 200 in week 1 to 220 in week 2, to 240 in week 3 and so on, until it is producing 600 in week N.
a. Find the value of N. **(2)**
The company then plans to continue to make 600 mobile phones each week.
b. Find the total number of phones that will be make in the first 52 weeks starting from and including week 1. **(2)**

6. Lewis played a game of space invaders. He scored points for each spaceship that he captured.
Lewis scored 140 points for capturing his first spaceship.
He scored 160 points for capturing his second spaceship, 180 points for capturing his third spaceship and so on.

a. Find the number of points that Lewis scored for capturing his 20^{th} spaceship. **(2)**

b. Find the total number of points Lewis scored for capturing his first 20 spaceships. **(3)**

Sian played an adventure game. She scored points for each dragon that she captured. The number of points that Sian scored for capturing each successive dragon form an arithmetic sequence.
Sian captured n dragons and the total number of points that she scored for capturing all n dragons was 8500.

Give thatn Sian scored 300 points for capturing her first dragon and then 700 for capturing nth dragon.

c. Find the value of n. **(3)**

7. An arithmetic progression has first term $\log_2 27$ and common difference $\log_2 x$.
a. Show that the fourth term can be written as $\log_2(27x^3)$ **(3)**
b. Given that the fourth term is 6, find the exact value of x. **(2)**

8a. Calculate the sum of all the even numbers from 2 to 100 inclusive, **(3)**

$$2 + 4 + 6 + \ldots. + 100$$

In the arithmetic series

$$k + 2k + 3k + \ldots.. + 100$$

k is a positive interger and k is a factor of 100.

b. Find, in terms of k, an expxression for the number of terms in this series. **(2)**

c. Show that the sum of this series is **(2)**

$$50 + \frac{5000}{k}$$

d. Find, in term of k, the 50th term of the arithmetic sequence

$$(2k + 1), (4k + 4), (6k + 7), \ldots.$$

Giving your answer in its simplest form. **(4)**

9. A 40 year building programme for new houses began in Oldtown in the year 1951 (Year 1) and finished in 1990 (Year 40). The number of house built each year form an arithmetic sequence with first term a and common difference d.

Given that 2400 new houses were built in 1960 and 600 new houses were built in 1990, find,

a. The value of d. **(3)**

b. The value of a. **(2)**

c. The total number of houses built in Oldtown over the 40 year period. **(3)**

Mark Scheme

1a.

$a = 21$ $21 + 2d = 27$	**M1**
$d = 3$	**M1**

1b.

$S_{40} = \frac{40}{2}[42 + (39 \times 3)]$	**M1**
$= 3180$	**M1**

2.

$n = 1$, first term $= 7 + 16 = 23$ $d = 7$	**M1**
$S_{35} = \frac{35}{2}[46 + (34 \times 7) = 4970$	**M1**

3.

$S_n = \frac{n}{2}[2a + (n - 1)d]$ $S_{10} = 400$ $400 = \frac{10}{2}[2a + 9d]$ $10a + 45d = 400$	**M1**
$S_{20} = 1400$ $1400 = \frac{20}{2}[2a + 19d]$ $20a + 190d = 1400$	**M1**
$10a + 45d = 400 \rightarrow 20a + 90d = 800$ $20a + 190d = 1400$	**M1**
$100d = 600$ $d = 6$	**M1**
$10a + 45(6) = 400$ $10a + 270 = 400$ $10a = 130$ $a = 13$	**M1**

4.

$b - a = (3a + 4) - b$	**M1**
$2b = 4a + 4$ $b = 2a + 2$	**M1**

5a.

$600 = 200 + (N - 1)(20)$	**M1**
$600 = 200 + 20N - 20$ $420 = 20N$ $N = 21$	**M1**

5b.

Total $= S_{21} + 600(31)$	**M1**
Total $= \frac{21}{2}(200 + 600) + 600(31)$ $= 27000$	**M1**

6a.

140, 160, 180, …. 20^{th} term = 140 + (20 – 1)(20)	**M1**
20^{th} term = 520	**M1**

6b.

Total points = $\frac{20}{2}(140 + 520)$	**M1**
= 6600	**M1**

6c.

1^{st} term = 300 nth term = 700 Sum = 8500	**M1**
$\frac{n}{2}(300 + 700) = 850$	**M1**
$500n = \frac{8500}{500}$ $n = 17$	**M1**

7a.

$U_n = a + (n - 1)d$	**M1**
$U_4 = \log_2 27 + 3\log_2 x$ $= \log_2 27 + \log_2 x^3$	**M1**
$U_4 = \log_2(27x^3)$	**M1**

7b.

$\log_2(27x^3) = 6$ $27x^3 = 2^6$	**M1**
$x^3 = \frac{2^6}{27}$ $x = \frac{4}{3}$	**M1**

8a.

$100 = 2 + (n - 1)2$ $100 = 2 + 2n - 2$ $n = 50$	**M1**
$2 + 4 + 6 + …. + 100 = S_{50}$	**M1**
$\frac{50}{2}(2 + 100) = 2550$	

8b.

Let n be the number of number of terms, $100 = k + (n - 1)k$	**M1**
$n = \frac{100}{k}$	**M1**

8c.

$k + 2k + 3k + … + 100 = \frac{1}{2}\left(\frac{100}{k}\right)[k + 100]$	**M1**
$= 50 + \frac{5000}{k}$	**M1**

8d.

$d = 4k + 4 – (2k + 1) = 4k + 4 – 2k - 1$ $d = 2k + 3$	**M1**
50th Term $= 2k + 1 + 49(2k + 3)$ $= 2k + 1 + 98k + 147$	**M1**
$= 100k + 148$	

9a.

$U_{10} = 2400$ $a + 9d = 24000$	**M1**
$U_{40} = 600$ $a + 39d = 600$	**M1**
30d = -1800 $d = -60$	**M1**

9b.

Substituting $d = -60$ into $a + 9d = 24000$ $a + 9(-60) = 2400$	**M1**
$a – 540 = 2400$ $a = 2940$	**M1**

9c.

$a = 2940$, $d = -60$ $S_{40} = \frac{40}{2}(2940 + 600)$	**M1**
$S_{40} = 70800$	**M1**

Sequences and Series

Part 3: Geometric Sequences

AS Level	A-Level	
Part 1a: Binomial Expansion 1	Part 1b: Binomial Expansion 2	Part 3: Geometric Sequences
	Part 2: Arithmetic Sequences	Part 4: Recurrence Relations

1. The first and fourth terms of a geometric series are 768 and −96 respectively.
a. Find the common ratio of the series. **(3)**
b. Find the tenth term of the series. **(1)**

2. The sum of the first four terms of a geometric series is 130 and its common ratio is $1\frac{1}{2}$
a. Find the first term of the series. **(2)**
b. Find the eighth term of the series. **(1)**
c. Find the least value of n for which the sum of the first n terms of the series is greater than 30 000. **(3)**

3. All the terms of a geometric series are positive. The sum of the first and second terms of the series is 10.8 and the sum of the third and fourth terms of the series is 43.2
a. Find the first term and common ratio of the series. **(4)**
b. Find the sum of the first 16 terms of the series. **(1)**

4. A geometric series has first term 80 and common ratio 0.2
a. Find the sum to infinity of the series. **(1)**
b. Find the difference between the sum to infinity of the series and the sum of the first six terms of the series. **(2)**

5. The sum, S_n , of the first n terms of a geometric series is given by $S_n = 2n - 1$.
a. Find the first term and the fifth term of the series. **(3)**
b. Find an expression for the nth term of the series. **(3)**

6a. Evulate $\sum_{r=3}^{10} 3^r$ **(1)**
b. Show that $\sum_{r=1}^{15}(2^r - 12r) = 64094$ **(3)**

7. When a ball is dropped onto a horizontal floor it bounces such that it reaches a maximum height of 60% of the height from which it was dropped.
a. Find the maximum height the ball reaches after its fourth bounce when it is initially dropped from 3 metres above the floor. **(1)**
b. Show that when the ball is dropped from a height of h metres above the floor it travels a total distance of $4h$ metres before coming to rest. **(3)**

8. A company predicts a yearly product profit of £120 000 in the year 2013. The company predicts that the yearly profit will rise each year by 5%. The predicted yearly profit forms a geometric sequence with common ratio 1.05.
a. Show that the predicted profit in the year 2016 in £138 915. **(1)**
b. Find the first year in which the yearly predicted profit exceeds £200,000. **(5)**
c. Find the total predicted profit for the years 2013 to 2023 inclusive, giving your answer to the nearest pound. **(3)**

9. The first three terms of a sequence are $2x$, $x + 4$, $2x - 7$ respectively.
a. Verify that when $x = 8$, the term form a geometric progression and find the sum to infinity in this case. **(4)**
b. Find the other possible value of x that also gives a geometric progressions. **(4)**

Mark Scheme

1a.

$a = 768$, $ar^3 = -96$	**M1**
$r^3 = -96 \div 768 = -\frac{1}{8}$	**M1**
$r = -\frac{1}{2}$	**M1**

1b.

$U_{10} = 768 \text{ x } (-0.5)^9 = 1.5$	**M1**

2a.

$\frac{a[1.5^4 - 1]}{1.5 - 1} = 130$	**M1**
$a = 130 \div \frac{65}{8} = 16$	**M1**

2b.

$U_8 = 16 \text{ x } (1.5)^7 = 273\frac{3}{8}$	**M1**

2c.

$\frac{16[1.5^n - 1]}{1.5 - 1} > 30000$ $1.5^n > 938.5$	**M1**
$n \log (\frac{3}{2}) > \log 938.5$ $n > \log \frac{938.5}{1.5}$	**M1**
$n > 16.9$ $n = 17$	**M1**

3a.

$a + ar = a(1 + r) = 10.8$	**M1**
$ar^2 + ar^3 = ar^2 (1 + r) = 43.2$ $r^2 = 43.2 \div 10.8 = 4$	**M1**
As all terms are positive, r must also be positive, therefore $r = 2$	**M1**
$a = 10.8 \div 3 = 3.6$	**M1**

3b.

$S_{16} = \frac{3.6(2^{16} - 1)}{2 - 1} = 235\ 926$	**M1**

4a.

$S_\infty = \frac{80}{1 - 0.2} = 100$	**M1**

4b.

$S_6 = \frac{80(1 - 0.2^6)}{1 - 0.2} = 99.9936$	**M1**
S_∞ - S6 = 0.0064	**M1**

5a.

$U_1 = S_1 = 2^1 - 1 = 1$	**M1**
$S_5 = 2^5 - 1 = 31$ $S_4 = 2^4 - 1 = 15$	**M1**
$U_5 = S_5 - S_4 = 31 - 15 = 16$	**M1**

5b.

$S_{n-1} = 2^{n-1} - 1$ $U_n = S_n - S_{n-1} = (2^n - 1)(2^{n-1} - 1)$	**M1**
$= 2^n - 2^{n-1}$ $= 2^{n-1}(2 - 1)$	**M1**
$= 2^{n-1}$	**M1**

6a.

$a = 27$, $r = 3$ $S_8 = \frac{27(3^8-1)}{3-1} = 88\ 560$	**M1**

6b.

$\sum_{r=1}^{15} 2^r \;\; a = 2, r = 2$ (geometric progression) $S_{15} = \frac{2(2^{15}-1)}{2-1} = 65\ 534$	**M1**
$\sum_{r=1}^{15} 12r \;\; a = 12, d = 12$ (arithmetic progression) $S_{15} = \frac{15}{2}[24 + (14 \times 12)] = 1440$	**M1**
$\sum_{r=1}^{15}(2^r - 12r) = 65\ 534 - 1440 = 64\ 094$	**M1**

7a.

After 4th bounce, reaches $3 \times 0.6^4 = 0.3888$m	**M1**

7b.

Total Distance = $h + 2[0.6 + 0.6^2h + 0.6^3h + \ldots]$	**M1**
$= h + 2 \times S_\infty$ of GP $a = 0.6h$ $r = 0.6$ $= h + \frac{2 \times 0.6h}{1-0.6}$	**M1**
$= h + 3h = 4h$ metres	**M1**

8a.

Predicted profit = $120\ 000\ (1.05)^3 = £138\ 915$	**M1**

8b.

$U_n > 200\ 000$ $120000(1.05)^{n-1} > 200\ 000$	**M1**
$\text{Log}(1.05)^{n-1} > \log\left(\frac{5}{3}\right)$ $(n-1)\log(1.05) > \log\left(\frac{5}{3}\right)$	**M1**
$(n-1)\log(1.05) > \log\left(\frac{5}{3}\right)$	**M1**
$n > \frac{\log\left(\frac{5}{3}\right)}{\log 1.05} + 1$ $n > 11.46$ therefore, $n = 12$	**M1**
Therefore year = 2013 + 11 = 2024	**M1**

8c.

Total profit = S_{11}	**M1**
$\frac{120000((1.05)^{11} - 1)}{1.05 - 1}$	**M1**
$S_{11} = £1704814.459\ldots = £1704814$ (to the nearest pound)	**M1**

9a.

When x = 8, terms are 16, 12, 9 since $\frac{12}{16} = \frac{3}{4}, \frac{9}{12} = \frac{3}{4}$	**M1**
Share a common ratio of $\frac{3}{4}$ there it is a geometric progression.	**M1**
$S_\infty = \frac{a}{1-r} = \frac{16}{1-0.75}$ = 64	**M1**
= 64	**M1**

9b.

As the series is a geometric progression, $$\frac{x+4}{2x} = \frac{2x-7}{x+4}$$	**M1**
$(x + 4)(x + 4) = 2x(2x – 7)$ $x^2 + 8x + 16 = 4x^2 – 14x$	**M1**
$3x^2 – 22x – 16 = 0$ $(3x + 2)(x – 8) = 0$	**M1**
$x = -\frac{2}{3}$, $x = 8$ Therefore $x = -\frac{2}{3}$	**M1**

Sequences and Series

Part 4: Reccurence Relations

Jethwa Maths

AS Level	A-Level	
Part 1a: Binomial Expansion 1	Part 1b: Binomial Expansion 2	Part 3: Geometric Sequences
	Part 2: Arithmetic Sequences	Part 4: Recurrence Relations

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1. For the following sequence, find U_2 and U_3 in terms of the constant k.

$$U_n = 4U_{n-1} + 3k \qquad n > 1,\ U_1 = 1$$ **(2)**

2. A sequence is given by the recurrence relation

$$U_n = \frac{1}{2}(k + 3u_{n-1}) \qquad n > 1,\ U_1 = 2$$

a. Find an expression for U_3 in terms of the constant k. **(2)**
Given that $U_3 = 7$,
b. find the value of k and the value of U_4. **(2)**

3. A sequence is defined by

$$U_{n+1} = U_n + c \qquad n \geq 1,\ U_1 = 2$$

where c is a constant. Given that $U_5 = 30$, find
a. The value of c, **(2)**
b. An expression for U_n in terms of n. **(2)**

4. A sequence of terms $\{tn\}$ is defined, for $n > 1$, by the recurrence relation

$$t_n = kt_{n-1} + 2,$$

where k is a constant. Given that $t_1 = 1.5$,
a. find expressions for t_2 and t_3 in terms of k. **(2)**
Given also that $t_3 = 12$,
b. find the possible values of k. **(2)**

5. $U_n = \cos(90n)$
State whether the period in increasing, decreasing or periodic (state the period). **(2)**

6. The sequence with reccurence relation $U_{k+1} = pU_k + q$, $U_1 = 5$, where p is a constant and $q = 10$, is periodic with order 2. Find the value of p. **(5)**

7. A sequence is defined by the reccurence relation
$a_{n+1} = \frac{1}{a^n}$, $a_1 = p$
a. Show that the sequence is periodic and state its order. **(3)**
b. Find $\sum_{r=1}^{1000} a_n$ in terms of p **(2)**

8. The sequence $U_1, U_2, U_3, \ldots$ is defined by $U_{n+1} = \frac{4}{2 - U_n}, U_1 = 1$
a. Show that this sequence is periodic, stating the period. **(3)**
b. Hence find $\sum_{n=1}^{50} U_n$ **(2)**

Mark Scheme

1.

$U_1 = 1$ $U_2 = 4(U_1) + 3k = 4(1) + 3k = 4 + 3k$	**M1**
$U_3 = 4(U_2) + 3k = 4(4 + 3k) + 3\text{k} = 16 + 15k$	**M1**

2a.

$U_2 = \frac{1}{2}(k + 6)$	**M1**
$U_3 = \frac{1}{2}[k + \frac{3}{2}(k + 6)] = \frac{1}{4}(5k + 18)$	**M1**

2b.

$\frac{1}{4}(5k + 18) = 7$ $k = 2$	**M1**
$U_4 = \frac{1}{2}(2 + 21) = 11\frac{1}{2}$	**M1**

3a.

$U_5 = 2 + 4C = 30$	**M1**
$C = 7$	**M1**

3b.

Sequence is 2, 9, 16, 23, 30, …	**M1**
Therefore $U_n = 7n - 5$	**M1**

4a.

$t_2 = 1.5k + 2$	**M1**
$t_3 = k(1.5k + 2) + 2 = 1.5k^2 + 2k + 2$	**M1**

4b.

$1.5k^2 + 2k + 2 = 12$ $3k^2 + 4k - 20 = 0$	**M1**
$(3k + 10)(k - 2) = 0$ $k = -3\frac{1}{3}, 2$	**M1**

5.

$U_1 = \cos 90 = 0$ $U_2 = \cos(180) = -1$ $U_3 = \cos(270) = 0$ $U_4 = \cos(360) = 1$ $U_5 = \cos(450) = 0$ $U_6 = \cos(540) = -1$ $U_7 = \cos(630) = 0$ $U_8 = \cos(720) = 1$	**M1**
The sequence is periodic with order 4.	**M1**

6.

$U_2 = 5p + 10$	**M1**
$U_3 = p(5p + 10) + 10 = 5p^2 + 10p + 10$	**M1**
Period 2 therefore, $U_1 = U_3 = 5$ $5p^2 + 10p + 10 = 5$	**M1**
$p^2 + 2p + 1 = 0$ $(p + 1)^2 = 0$	**M1**
$p = -1$	**M1**

7a.

$a_1 = p$ $a_2 = \frac{1}{p}$	**M1**
$a_3 = \frac{1}{\frac{1}{p}} = 1 \times \frac{p}{1} = p$ $a_4 = \frac{1}{p}$	**M1**
Sequence has a periodic order of 2.	**M1**

7b.

$\sum_{r=1}^{1000} a_n = \frac{1000}{2}(p + \frac{1}{p})$	**M1**
$= 500\ (p + \frac{1}{p})$	**M1**

8a.

$U_2 = \frac{4}{2-1} = \frac{4}{1} = 4$	**M1**
$U_3 = \frac{4}{2-4} = \frac{4}{-2} = -2$ $U_4 = \frac{4}{2--2} = \frac{4}{4} = 1$ $U_5 = \frac{4}{2-1} = \frac{4}{1} = 4$	**M1**
As $U_1 = U_4$ and $U_2 = U_5$, sequence is periodic with order 3	**M1**

8b.

$U_1 + U_2 + U_3 = 1 + 4 + -2 = 3$ First 50 terms are made of $16(U_1 + U_2 + U_3) + U_1 + U_2$	**M1**
$\sum_{n=1}^{50} U_n = (16 \times 3) + 1 + 4 = 53$	**M1**