A-Level Unit Test

# PROOF

Jethwa Maths

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1. If n is a number between 2 and 5, then $n^3 + n$ is has a factor of 5. True or false? Prove your claim. (3)

2. Show that $x^2 - 8x + 17 > 0$ for all real values of x. (3)

3. Prove that there is an infinite amount of prime numbers (5)

4. Give a counter-example to prove that each of the following statements are false:

a) If $a^2 - b^2 > 0$, where a and b are real, then $a - b > 0$. (3)
b) There are no prime numbers divisible by 7. (1)
c) If x and y are irrational and $x \neq y$, then xy is irrational. (2)
d) For all real values of x, $\cos (90 - |x|)° = \sin x°$. (2)

5. Use proof by contradiction to prove each of the following statements.
a) If $n^3$ is odd, where n is a positive integer, then n is odd. (3)
b) If x is irrational, then x is irrational. (3)
c) If a, b and c are integers and bc is not divisible by a, then b is not divisible by a. (3)

6. a) Prove that if 2 = p where p and q are integers, then p must be even. (3)
b) Use proof by contradiction to prove that 2 is irrational. (4)

7. Prove that the sum of three consecutive integers is divisible by 3. (3)

**Total marks: 38**

# Mark Scheme

1.

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|---|---|
| Using proof by exhaustion<br>n = 2, $(2)^3 + 2 = 10$ (factor of 5) | M1 |
| n = 3, $(3)^3 + 3 = 30$ (factor of 5)<br>n = 4, $(4)^3 + 4 = 68$ ( not factor of 5) | M2 |
| When n = 4, the expression is not a factor of 5 and therefore the claim is false. | A1 |

2.

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|---|---|
| $x^2 - 8x + 17 = (x-4)^2 - 16 + 17 = (x-4)^2 + 1$ | M1 |
| $(x-4)^2$ will always give a positive value | M2 |
| Therefore, a positive value plus a positive value will always give a value above 0. | A1 |

3.

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|---|---|
| Proof by contradiction | M1 |
| Assuming there are a finite number of prime numbers, that we write as:<br>$p_1, p_2, p_3, \ldots, p_n$ | M2 |
| And we define a new number as:<br>$m = p_1 \times p_2 \times p_3 \times \ldots\ p_n + 1$ | M3 |
| As we are saying that there are no other primer numbrs than the original list defined, then m should not be a prime number and therefore divisible by $p_n$ | M4 |
| However, when we do this, we are left with a remainder of 1, and as there are no integers that divide by 1, then m must also be a prime number.<br>This is a contradiction. Hence, there are infinitely many primer numbers | A1 |

4a.

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|---|---|
| a = -2, b = 1 | M1 |
| $a^2 - b^2 = 4 - 1 = 3$ therefore, $a^2 - b^2 > 0$ | M2 |
| $a - b = -2 - 1 = -3$ therefore, $a - b < 0$ | A1 |

4b.

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|---|---|
| 7 is a prime and divisible by 7 | M1 |

4c.

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|---|---|
| $x = \sqrt{2}$, $y = 2\sqrt{2}$ (both irrational) | M1 |
| xy = 4 (rational) | M2 |

4d.

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|---|---|
| x = -90 | M1 |
| $\cos(90 - \|x\|) = \cos 0 = 1$<br>$\sin x = \sin(-90) = -1$ | M2 |

5a.

| | |
|---|---|
| Assuming $n^3$ is odd and n is even, | M1 |
| when n is even, n = 2m<br>Therefore, $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$ | M2 |
| $2(4m^3)$ means $n^3$ is even and therefore is a contradiction and therefore n is odd. | A1 |

5b.

| | |
|---|---|
| Assuming x is irrational and $\sqrt{x}$ is rational | M1 |
| $\sqrt{x} = \frac{p}{q}$<br>$x = \frac{p^2}{q^2}$<br>$p^2$ and $q^2$ are both rational and therefore x is rational. | M2 |
| This is a contradiction and therefore $\sqrt{x}$ is irrational. | A1 |

5c.

| | |
|---|---|
| Assuming that bc is not divisible by a and b and is divisible by a, | M1 |
| If b is divisible by a then b = ka<br>bc = kac (which is divisible by a | M2 |
| This is a contradiction and therefore b is not divisible by a. | A1 |

6a.

| | |
|---|---|
| $\sqrt{2} = \frac{p}{q}$ | M1 |
| $2 = \frac{p^2}{q^2}$<br>$p^2 = 2q^2$ | M2 |
| Therefore, $p^2$ is even, therefore p is even. | A1 |

6b.

| | |
|---|---|
| Assume $\sqrt{2}$ is rational, $\sqrt{2} = \frac{p}{q}$ and p is even | M1 |
| p = 2n<br>$p^2 = 2q^2$<br>$(2n)^2 = 2q^2$<br>$q^2 = 2n^2$ | M2 |
| Therefore $q^2$ is even, and q is even | M3 |
| This is a contradiction and therefore $\sqrt{2}$ is irrational | A1 |

7.

| | |
|---|---|
| Three consecutive integers, n, n + 1, n + 2 | M1 |
| Their sum: n + n + 1 + n + 2 = 3n + 3 = 3(n + 1) | M2 |
| As the sum has a factor of 3, the sum of three consecutive integers will is divisible by 3. | A1 |