Full Compilation
PROOF

AS Level
Proof by Deduction
Proof by Exhaustion
Proof by Counter Example

A-Level
Proof by Contradiction

Jethwa Maths

1. Prove that the sum of two consecutive odd numbers is a multiple of 4 (3)

2. Prove that $(2n + 3)^2 – (2n – 3)^2$ is a multiple of 6 for all values of n (3)

3. N is an odd integer that is not divisible by 3. Prove that N^2 is not a multiple of 3. (3)

4. If n is a positive integer, then $n^3 + n$ is even. (2)

5. Prove by exhaustion that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for all positive integers from 1 to 6 inclusive (3)

6. Use a counter example to show that the following statement is false: $n^2 – n – 1$ is a prime number, for $3 \leq n \leq 10$ (2)

7. Prove, by counter example, that the statement “if a is rational and b is irrational, then $\log_a b$ is irrational” is false. (2)

8. Using proof by contradiction prove that if $(n^2 + 2n)$ is even, where n is an integer, then n is even. (4)

9. Use proof by contradiction to prove that there are no positive integers, x and y, such that (4)

10. Use proof by contradiction to prove that $\log_2 3$ is irrational. (4)

11. Prove that if the equation: $k \cos x – \operatorname{cosec} x = 0$, where k is a constant, has real solutions, then $|k| \geq 2$. (5)

12. For each statement, either prove that is it true or find a counter-example to prove that it is false.

a) In m and n are consecutive odd integers, then (m + n) is divisible by 4. (3)

b) For all real values of x, $\cos x \leq 1 + \sin x$ (2)

<u>Mark Scheme</u>

1.

Assume two consecutive odd numbers are: $2n + 1$, $2n + 3$.	M1
Therefore, $(2n + 1) + (2n + 3) = 4n + 4$	M1
$= 4(n + 1)$. Therefore the sum of two consecutive odd numbers if are multiples of 4.	A1

2.

$(2n + 3)^2 – (2n – 3)^2 = (2n + 3)(2n + 3) – (2n - 3)(2n - 3)$	M1
$= (4n^2 + 6n + 6n + 9) – (4n^2 – 6n – 6n + 9)$ $= (4n^2 + 12n + 9) – (4n^2 – 12n + 9)$ $= 4n^2 + 12n + 9 - 4n^2 – 12n – 9$ $= 24n$	M1
$= 6(4n)$. Therefore, the expression is a multiple of 6 for all values of n.	A1

3.

If N is not a multiple of 3, then $n = 3k + 1$	M1
$\therefore$ $N^2 = (3k + 1)^2 = (3k + 1)(3k + 1)$ $= 9k^2 + 6k + 1$ $= 3(3k^2 + 2k) + 1$.	M1
As the first term will be a multiple of 3, adding 1, means N^2 will never be a multiple of 3.	A1

4.

$n^3 + n = n(n^2 + 1)$.	M1
If n is even, then n multiplied by $(n^2 + 1)$ must be even as even multiplied by anything, must make an even answer.	M1

5.

$n = 1 : \frac{(1)(2)}{2} = 1$ $n = 2 : \frac{(2)(3)}{2} = 3$ $n = 3 : \frac{(3)(4)}{2} = 6$ $n = 4 : \frac{(4)(5)}{2} = 10$ $n = 5 : \frac{(5)(6)}{2} = 15$ $n = 6 : \frac{(6)(7)}{2} = 21$	M1 (at least 2 n values) M1 (n values from 1 – 6)
As statement is true for all values n = 1 to 6, then by 'exhaustion' it has been proved.	A1

6.

Using trial and error:	M1
$n = 3 : 3^2 - 3 - 1 = 5$ (prime) $n = 5 : 5^2 - 5 - 1 = 19$ (prime) $n = 8 : 8^2 - 8 - 1 = 55$ (not prime)	M1

7.

If a = 2 (rational), b = $\sqrt{2}$ (irrational)	M1
$\log_a b = \log_2 \sqrt{2} = \frac{1}{2}$. Therefore the statement is not false.	M1

8.

Assume $n^2 + 2n$ is even, where n is odd. If n is odd, then $n = 2m + 1$	M1
$n^2 + 2n = (2m + 1)^2 + 2(2m + 1) = 4m^2 + 8m + 3$	M1
$= 2(2m^2 + 4m + 1) + 1$ (odd)	M1
This is therefore a contradiction and therefore n must be even.	A1

9.

$x^2 - y^2 = (x + y)(x - y) = 1$	M1
Therefore x + y = 1 or x – y = 1.	M1
Solving for x and y, 2x = 2 x = 1, y = 0	M1
This is therefore a contradiction, therefore no positive integer solutions.	A1

10.

Assume, $\text{Log}_2 3 = \frac{p}{q}$ is rational.	M1
$\text{Log}_2 3 = \frac{p}{q}$ is equivalent to $2^{\frac{p}{q}} = 3$	M1
$2^{\frac{p}{q}} = 3$ $(2^{\frac{p}{q}})^q = 3^q$ $2^p = 3^q$	M1
As 2 and 3 are co-prime, p = q = o. This is therefore a contradictions and $\log_2 3$ is irrational	M1

11.

$k \cos x - \text{cosec } x = 0$	M1
$k \cos x = \frac{1}{\sin x}$	M1
$k \sin x \cos x = 1$ $\frac{1}{2} k \sin 2x = 1$ $\sin 2x = \frac{2}{k}$	M1
$\left\|\frac{2}{k}\right\| \leq 1$	M1
$\|k\| \geq 2$	A1

12a.

a) Assuming the statement is true and m and n are consecutive integers, m = 2k + 1, n = 2k + 3	M1
m + n = 2k + 1 + 2k + 3 = 4k + 4 = 4(k + 1)	M1
Therefore, m + n is divisible by 4	A1

12b.

False; $x = \frac{5\pi}{3}$ cos x = $\frac{1}{2}$ and 1 + sin x = 1 - $\frac{\sqrt{3}}{2}$	M1
$1 - \frac{\sqrt{3}}{2} < \frac{1}{2}$, therefore false.	A1