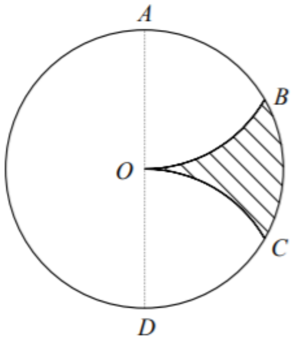
Practice Exam Paper F
Time: 2 Hours

P1 P2

1. The figure shows a circle of radius r and centre O in which AD is a diameter.



The points B and C lie on the circle such that OB and OC are arcs of circles of radius r with centres A and D respectively.

Show that the area of the shaded region OBC is $\frac{1}{6}r^2(3\sqrt{3} - \pi)$ **(6)**

(Total Marks: 6)

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2. During one day, a biological culure is allowed to grow under controlled conditions. At 8 a.m. the culture is estimated to contain 20 000 bacteria. A model of the growth of the culture assumes that t hours after 8 a.m., the number of bacteria present, N, is given by,

$$N = 20\,000 \times (1.06)^t$$

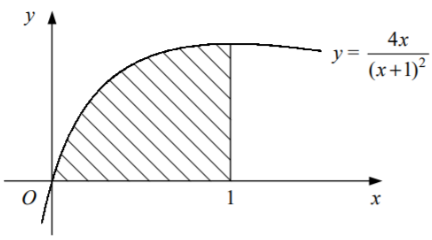
Using this model,

a. Find the number of bacteria present at 11 a.m **(2)**

b. Find, to the nearest minute, the time when the initial number of bacteria will have doubled. **(4)**

(Total Marks: 6)

3.



The figure shows the curve with equation $y = \frac{4}{(x+1)^2}$

The shaded region is bounded by the curve, the x-axis and the line $x = 1$.

a. Use the trapezium rule with four intervals of equal width to find an estimate for the area of the shaded region. **(5)**

b. State, with a reason, whether your answer to part (a) is an under-estimate or an over-estimate of the true area. **(2)**

(Total Marks: 7)

4. The first three terms in the expansion in descending powers of x of

$$(x + \frac{k}{x^2})^{15}$$

where k is a constant, are $x^{15} + 30x^{12} + Ax^9$

a. Find the values of k and A. **(5)**

b. Find the value of the term independent of x in the expansion. **(3)**

(Total Marks: 8)

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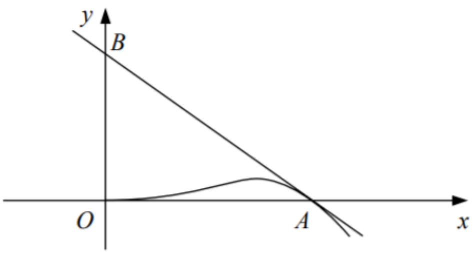
5. A curve has the equation $2 \sin 2x - \tan y = 0$.

a. Find an expression for $\frac{dy}{dx}$ in its simplest form in terms of x and y **(5)**

b. Show that the tangent to the curve at the point ($\frac{\pi}{6}$, $\frac{\pi}{3}$) has the equation $y = \frac{1}{2}x + \frac{\pi}{4}$ **(3)**

(Total Marks: 8)

6.



The figure shows the curve with parametric equations

$$x = a\sqrt{t}$$
$$y = at(1 - t)$$
$$t \geq 0$$

where a is a positive constant

a. Find $\frac{dy}{dx}$ in terms of t. **(3)**

The curve meets the x-axis at the origin, O, and at the point A. The tangent to the curve at A meets the y-axis at the point B as shown.

b. Show that the area of triangle OAB is a^2 **(6)**

(Total Marks: 9)

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7. The gradient at any point (x, y) on a curve is proportional to $\sqrt{y}$.

Given that the curve passes through the point with coordinates (0, 4),

a. Show that the equation of the curve can be written in the form

$$2\sqrt{y} = kx + 4$$

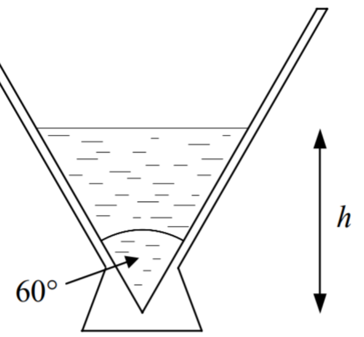
where k is a positive constant. **(5)**

Given also that the curve passes through the point with coordinates (2, 9),

b. Find the equation of the curve in the form $y = f(x)$. **(4)**

(Total Marks: 9)

8.



The figure shows a vertical cross-section of a vase.

The inside of the vase is in the shape of a right-circular cone with the angle between the sides in the cross-section being 60°. When the depth of water in the vase is h cm, the volume of water in the vase is V cm^3.

a. Show that $V = \frac{1}{9}\pi h^3$ **(3)**

The vase is initially empty and water is poured in at a constant rate of 120 $cm^3\ s^{-1}$

b. Find, to 2 decimal places, the rate at which h is increasing

i. When $h = 6$,

ii. After water has been poured in for 8 seconds. **(7)**

(Total Marks: 10)

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9. $f(x) = \frac{x(3x-7)}{(1-x)(1-3x)}, |x| < \frac{1}{3}$

a. Find the values of the constants A, B and C such that

$f(x) = A + \frac{B}{1-x} + \frac{C}{1-3x}$ **(4)**

b. Evaluate,

$$\int_0^{\frac{1}{4}} f(x)\, dx$$

giving your answer in the form $p + \ln q$, where p and q are rational **(5)**

c. Find the series expansion of $f(x)$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient. **(5)**

(Total Marks: 14)

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10. The curve C has the equation $y = 2e^x - 6 \ln x$ and passes through the point P with x-coordinate 1.

a. Find an equation for the tangent to C at P. **(4)**

The tangent to C at P meets the coordinate axes at the points Q and R.

b. Show that the area of triangle OQR, where O is the origin, is $\frac{9}{3-e}$. **(4)**

(Total Marks: 8)

11a. Sketch on the same diagram the graphs of $y = |x| - a$ and $y = |3x + 5a|$, where a is a positive constant.

Show on your diagram the coordinates of any points where each graph meets the coordinate axes. **(5)**

b. Solve the equation, $|x| - a = |3x + 5a|$ **(4)**

(Total Marks: 9)

12a. Use the identity,

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

to prove that,

$$\cos x \equiv 1 - 2\sin^2 \frac{x}{2}$$ **(3)**

b. Prove that, for $\sin x \neq 0$,

$$\frac{1-\cos x}{\sin x} \equiv \tan\frac{x}{2}$$

(3)

c. Find the values of x in the interval $0 \leq x \leq 360°$ for which,

$$\frac{1 - \cos x}{\sin x} = 2\sec^2\frac{x}{2} - 5$$

giving your answers to 1 decimal place where appropriate. **(6)**

(Total Marks: 12)

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13. A curve has the equation $y = (2x + 3)\,e^{-x}$

a. Find the exact coordinates of the stationary point of the curve. **(4)**

The curve crosses the y-axis at the point P.

b. Find an equation for the normal to the curve at P **(2)**

The normal to the curve at P meets the curve again at Q

c. Show that the x-coordinate of Q lies in the interval $[-2, -1]$ **(3)**

d. Use the iterative formula,

$$x_{n+1} = \frac{3 - 3e^{x_n}}{e^{x_n} - 2}$$

with $x_0 = -1$, to find, x_1, x_2, x_3, x_4. Give the value of x_4 to 2 decimal places. **(3)**

e. Show that your value for x_4 is the x-coordinate of Q correct to 2 decimal places. **(2)**

(Total Marks: 14)

Total Marks: 120

Mark Scheme

1	area of segment = $(\frac{1}{2} \times r^2 \times \frac{\pi}{3}) - (\frac{1}{2} \times r^2 \times \sin\frac{\pi}{3})$	**B1** **M2**
	$= \frac{1}{6}r^2\pi - \frac{1}{4}r^2\sqrt{3}$	**A1**
	Shaded area = $\frac{1}{6}r^2\pi - 2(\frac{1}{6}r^2\pi - \frac{1}{4}r^2\sqrt{3})$	**M1**
	$= \frac{1}{6}r^2\pi - \frac{1}{3}r^2\pi + \frac{1}{2}r^2\sqrt{3}$ $= \frac{1}{2}r^2\sqrt{3} - \frac{1}{6}r^2\pi = \frac{1}{6}r^2(3\sqrt{3} - \pi)$	**A1**

2a	11 a.m. ∴ $t = 3$ $N = 20\,000 \times (1.06)^3 = 23820$ (nearest unit)	**B1** **M2**
2b	$40\,000 = 20\,000 \times (1.06)^t$ $(1.06)^t = 2$	**M1**
	$T = \frac{\log 2}{\log 1.06} = 11.8957$	**M1** **A1**
	11.8957 hours = 11 hours 54 mins ∴ 7.54 p.m.	**A1**

3a

x	0	0.25	0.5	0.75	1
$\frac{4x}{(x+1)^2}$	0	0.64	0.8889	0.9796	1

3a	(table above)	**M1** **A1**
	Area = $\frac{1}{2} \times 0.25 \times [0 + 1 + 2(0.64 + 0.8889 + 0.9796)]$	**B1** **M1**
	= 0.752 (3 s.f)	**A1**
3b	Under-estimate	**B1**
	The curve passes above the top edge of each trapezium	**B1**

4a	$(x + \frac{k}{x^2})^{15} = x^{15} + 15(x^{14})\left(\frac{k}{x^2}\right) + \binom{15}{2}(x^{13})(\frac{k}{x^2})^2 + \cdots$	**M1** **A1**
	Therefore, $15k = 30$	**M1**
	$k = 2$	**A1**
	$A = \frac{15\times14}{2} \times k^2 = 420$	**A1**
4b	$(x + \frac{2}{x^2})^{15} = \cdots + \binom{15}{5}(x^{10})(\frac{2}{x^2})^5 + \cdots$	**M1** **A1**
	Term independent of $x = \frac{15\times14\times13\times12\times11}{5\times4\times3\times2} \times 32 = 96096$	**A1**

5a	$4\cos 2x - \sec^2 y\,\frac{dy}{dx} = 0$	**M1** **A2**
	$\frac{dy}{dx} = 4\cos 2x \cos^2 y$	**M1** **A1**
5b	Gradient = $4 \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{2}$	**B1**
	$y - \frac{\pi}{3} = \frac{1}{2}(x - \frac{\pi}{6})$	**M1**
	$y - \frac{\pi}{3} = \frac{1}{2}x - \frac{\pi}{12}$ $y = \frac{1}{2}x + \frac{\pi}{4}$	**A1**

6a	$\frac{dx}{dt} = \frac{1}{2}at^{-\frac{1}{2}}$ $\frac{dy}{dt} = a(1 - 2t)$	**M1**

	$\frac{dy}{dx} = \frac{a(1-2t)}{\frac{1}{2}at^{-\frac{1}{2}}} = 2\sqrt{t}(1-2t)$	**M1** **A1**
6b	$y = 0 \Rightarrow t = 0$ (at O) or 1 (at A)	**B1**
	$t = 1$, $x = a$, $y = 0$, grad $= -2$	**M1**
	$y - 0 = -2(x - a)$	**A1**
	at B, $x = 0 \therefore y = 2a$	**M1**
	area = $\frac{1}{2} \times a \times 2a = a^2$	**M1** **A1**

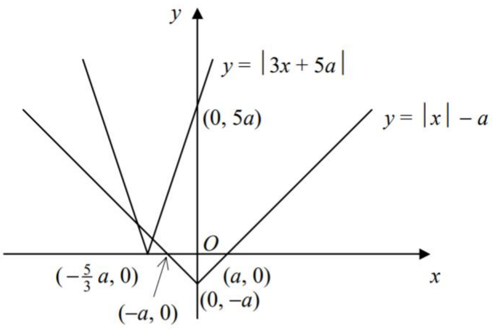
7a	$\frac{dy}{dx} = k\sqrt{y}$ $\int y^{-\frac{1}{2}}\, dy = \int k\ dx$	**M1**
	$2y^{\frac{1}{2}} = kx + c$	**M1** **A1**
	$(0, 4) \rightarrow 4 = c$	**M1**
	$2\sqrt{y} = kx + 4$	**A1**
7b	$(2, 9) \rightarrow 6 = 2k + 4$ $k = 1$	**M1** **A1**
	$2\sqrt{y} = x + 4$ $\sqrt{y} = \frac{1}{2}(x + 4)$	**M1**
	$y = \frac{1}{4}(x + 4)^2$	**A1**

8a	let radius = r Therefore $\tan 30 = \frac{1}{\sqrt{3}} = \frac{r}{h}$ $r = \frac{h}{\sqrt{3}}$	**M1**
	$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h \times \frac{h^2}{3} = \frac{1}{9}\pi h^3$	**M1** **A1**
8bi	$\frac{dV}{dt} = 120$ $\frac{dV}{dh} = \frac{1}{3}\pi h^2$	**B1**
	$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $120 = \frac{1}{3}\pi h^2 \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{360}{\pi h^2}$	**M1** **A1**
	When $h = 6$, $\frac{dh}{dt} = 3.18 cm\ s^{-1}$ (2 d.p)	**M1** **A1**
8bii	$V = 8 \times 120 = 960 = \frac{1}{9}\pi h^3$ $h = \sqrt[3]{\frac{9 \times 960}{\pi}} = 14.011$	**M1**
	$\frac{dh}{dt} = 0.58\ cm\ s^{-1}$ (2 d.p)	**A1**

9a	$x(3x - 7) \equiv A(1 - x)(1 - 3x) + B(1 - 3x) + C(1 - x)$	**M1**
	$x = 1 \Rightarrow -4 = -2B \Rightarrow B = 2$	**A1**
	$x = \frac{1}{3} \Rightarrow -2 = \frac{2}{3}C \Rightarrow C = -3$	**A1**
	coeffs $x^2 \Rightarrow 3 = 3A \Rightarrow A = 1$	**A1**
9b	$\int_0^{\frac{1}{4}} \left(1 + \frac{2}{1-x} - \frac{3}{1-3x}\right) dx = [x - 2\ln\lvert 1 - x\rvert + \ln\lvert 1 - 3x\rvert]_0^{\frac{1}{4}}$	**M1** **A1**
	$= (\frac{1}{4} - 2\ln\frac{3}{4} + \ln\frac{1}{4}) - (0)$	**M1**
	$= \frac{1}{4} + \ln\frac{16}{9} + \ln\frac{1}{4}$	**M1** **A1**

	$= \frac{1}{4} + \ln\frac{4}{9}$	
9c	$f(x) = 1 + 2(1 - x)^{-1} - 3(1 - 3x)^{-1}$ $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$	**B1**
	$(1 - 3x)^{-1} = 1 + 3x + (3x)^2 + (3x)^3 + \ldots = 1 + 3x + 9x^2 + 27x^3 + \ldots$	**M1** **A1**
	$\therefore f(x) = 1 + 2(1 + x + x^2 + x^3 + \ldots) - 3(1 + 3x + 9x^2 + 27x^3 + \ldots)$	**M1**
	$= -7x - 25x^2 - 79x^3 + \ldots$	**A1**

10a	$\frac{dy}{dx} = 2e^x - \frac{6}{x}$	**M1**
	$x = 1, y = 2e$, grad $= 2e - 6$	**A1**
	$y - 2e = (2e - 6)(x - 1)$	**M1** **A1**
10b	$x = 0 \Rightarrow y = 6$ $y = 0 \Rightarrow (2e - 6)x + 6 = 0$ $x = -\frac{6}{2e-6} = \frac{3}{3-e}$	**M1** **A1**
	Area $= \frac{1}{2} \times 6 \times \frac{3}{3-e} = \frac{9}{3-e}$	**M1** **A1**

11a	 	**B3** **B3**
11b	$-x - a = 3x + 5a \Rightarrow x = -\frac{3}{2}a$	**M1** **A1**
	$-x - a = -(3x + 5a) \Rightarrow x = -2a$ $x = -2a, -\frac{3}{2}a$	**M1** **A1**

12a	$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ Let $A = B = \frac{x}{2}$ $\cos x \equiv \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$	**M1**
	$\cos x \equiv (1 - \sin^2\frac{x}{2}) - \sin^2\frac{x}{2}$	**M1**
	$\cos x \equiv 1 - 2\sin^2\frac{x}{2}$	**A1**
12b	LHS $\equiv \frac{1-(1-2\sin^2\frac{x}{2})}{2\sin\frac{x}{2}\cos\frac{x}{2}}$	**M1**
	$\equiv \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \equiv \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \equiv \tan\frac{x}{2} \equiv$ RHS	**M1** **A1**
12c	$\tan\frac{x}{2} = 2\sec^2\frac{x}{2} - 5$ $\tan\frac{x}{2} = 2(1 + \tan^2\frac{x}{2}) - 5$	**M1**
	$2\tan^2\frac{x}{2} - \tan\frac{x}{2} - 3 = 0$ $(2\tan\frac{x}{2} - 3)(\tan\frac{x}{2} + 1) = 0$	**M1**
	$\tan\frac{x}{2} = -1$ $\tan\frac{x}{2} = \frac{3}{2}$	**A1**
	$\frac{x}{2} = 135$	**B1**

	$\frac{x}{2} = 56.310$	
	$x = 112.6^{\circ}$ (1 d.p) or 270°	**A2**

13a	$\frac{dy}{dx} = 2 \times e^{-x} + (2x + 3) \times (-e^{-x}) = -(2x + 1)e^{-x}$	**M1** **A1**
	S.P: $-(2x + 1)e^{-x} = 0$ $x = -\frac{1}{2}$ Therefore, $(-\frac{1}{2}, 2e^{\frac{1}{2}})$	**M1** **A1**
13b	$x = 0$, $y = 3$, grad $= -1$, grad of normal $= 1$	**M1**
	$y = x + 3$	**A1**
13c	$x + 3 = (2x + 3)e^{-x}$ $x + 3 - (2x + 3)e^{-x} = 0$	**M1**
	let $f(x) = x + 3 - (2x + 3)e^{-x}$ $f(-2) = 8.4$, $f(-1) = -0.72$	**M1**
	sign change, $f(x)$ continuous $\therefore$ root	**A1**
13d	$x_1 = -1.1619$, $x_2 = -1.2218$, $x_3 = -1.2408$, $x_4 = -1.2465 = -1.25$ (2dp)	**M1** **A2**
13e	$f(-1.255) = 0.026$, $f(-1.245) = -0.016$	**M1**
	sign change, $f(x)$ continuous $\therefore$ root	**A1**

Topic List

Q1	Radians
Q2	Exponential modelling
Q3	Trapezium rule
Q4	Binomial expansion
Q5	Implicit differentiation
Q6	Parametric equations
Q7	Differential equations
Q8	Connected rates
Q9	Partial fractions and binomial series
Q10	Differentiation
Q11	Sketching modulus graphs
Q12	Trigonometry
Q13	Differentiation, numerical methods