Numerical Methods

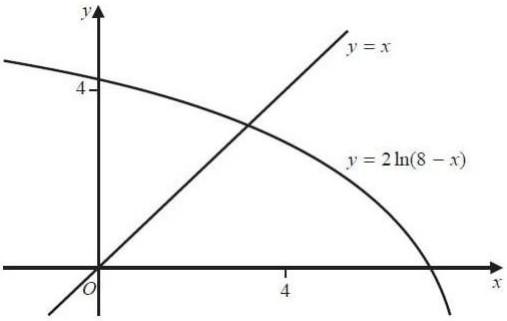
Pt. 1: Iteration

A-Level

Pt.1 : Iteration

Pt. 2: Newton-Raphson Method

1. The curve with equation $y = 2\ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.
a. Show that $3 < \alpha < 4$. **(2)**



The figure shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$. A students uses the iteration formula $x_{n+1} = 2\ln(8 - x_n)$ in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

b. Determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer. **(2)**

2. Figure 2 shows a sketch of part of the curve with equation,

$$y = 2\cos\left(\tfrac{1}{2}x^2\right) + x^3 - 3x - 2$$

The curve crosses the x-acis at the point Q and has a minimum turning point at R.
a. Show that the x-coordinate of Q lies between 2.1 and 2.2. **(2)**
b. Show that the x-coordinate of R is a solution of the equation. **(4)**

$$x = \sqrt{1 + \tfrac{2}{3}x\sin\left(\tfrac{1}{2}x^2\right)}$$

Using the iterative formula:

$$x_{n+1} = \sqrt{1 + \tfrac{2}{3}x_n\sin\left(\tfrac{1}{2}{x_n}^2\right)}$$

c. Find the values of x_1 and x_2 to 3 decimal places. **(2)**

3. $f(x) = 25x^2e^{2x} - 16$
a. Using calculus, find the exact coordinates of the turning points on the curve with equation $y = f(x)$. **(5)**
b. Show that the equation $f(x) = 0$, can be written as $x = = \pm\frac{4}{5}e^{-x}$ **(1)**
The equation $f(x) = 0$ has a root α, where $\alpha = 0.5$ to 1 decimal place.
c. Starting with $x_0 = 0.5$, use the iteration formula $x_{n+1} = \frac{4}{5}e^{-x_n}$ to calculate the values of x_1, x_2 and x_3 giving your answers to 3 decimal places. **(3)**
d. Give an accurate estimate for α to 2 decimal places, and justify your answer. **(2)**

Mark Scheme

1a.

At the point of intersection, $2\ln(8 - x) = x$ $2\ln(8 - x) - x = 0$	**M1**
f(3) = 0.2188 f(4) = -1.2274 There is a root α in the interval [3, 4] since there is a change in sign and the curve is continuous over the interval.	**M1**

1b.

Demonstration of cobweb graph	**M1**
The cobweb graph sprials inwards towards α, threfeore the iteration formula can be used as an approximation for α.	**M1**

2a.

When $x = 2.1$, $y = -0.2240\ldots$ When $x = 2.2$, $y = 0.5464\ldots$	**M1**
Therefore x-coordainte of Q lies between 2.1 and 2.2 since there is a change of sign and the curve is continuous.	**M1**

2b.

Attempt to differentiate $y = 2\cos(\frac{1}{2}x^2) + x^3 - 3x - 2$	**M1**
$\frac{dy}{dx} = 2\left[-\sin\left(\frac{1}{2}x^2\right)\right](x) + 3x^2 - 3$	**M1**
At R, $\frac{dy}{dx} = 0$ $3x^2 = 3 + 2x(\sin\frac{1}{2}x^2)$	**M1**
$x^2 = 1 + \frac{2}{3}x\sin(\frac{1}{2}x^2)$ $x = \sqrt{1 + \frac{2}{3}x\sin(\frac{1}{2}x^2)}$	**M1**

2c.

$x_1 = 1.28383\ldots = 1.284$ (3 d.p)	**M1**
$x_2 = 1.2760\ldots = 1.2760$ (3 d.p)	**M1**

3a.

$y = 25x^2e^{2x} - 16$ $\frac{dy}{dx} = 25x^2(2e^{2x}) + (e^{2x})(50x)$	**M1**
$\frac{dy}{dx} = 50x^2e^{2x} + 50xe^{2x} = 50xe^{2x}(x + 1)$	**M1**
At turning point, $\frac{dy}{dx} = 0$, $50xe^{2x}(x + 1) = 0$ $x = 0$ $x = -1$	**M1**
When $x = 0$, $y = 25(0)e^0 - 16 = -16$	**M1**
When $x = -1$, $y = 25(-1)^2e^{-2} - 16 = 25e^{-2} - 16$	**M1**
Turning points: $(0, -16)$, $(-1, 25e^{-2} - 16)$	

3b.

When f(x) = 0, $25x^2e^{2x} - 16 = 0$ $x^2 = \frac{16}{25e^{2x}}$ $x = \pm\frac{4}{5}e^{-x}$	**M1**

3c.

$x_0 = 0.5$ $x_1 = 0.4852\ldots = 0.485$ (to 3d.p)	**M1**
$x_2 = 0.4924\ldots = 0.492$ (to 3d.p)	**M1**
$x_3 = 0.4889\ldots = 0.489$ (to 3d.p)	**M1**

3d.

$\alpha = 0.49$ f(0.485) = -0.4872… f(0.495) = 0.4854….	**M1**
As there is a change in sign and the function is continuous, $\alpha = 0.49$ (to 2 d.p)	**M1**

Mark Scheme

1.

	M1
	M1