A-level Further Mathematics Further Pure 1 and 2
Paper J

1i.

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

a. Describe fully the single transformation represented by the matrix **A.** **[2]**

The matrix **B** represents a stretch, scale factor 3, parallel to the x-axis.

b. Find the matrix **B** **[2]**

ii.

$$\mathbf{M} = \begin{pmatrix} -4 & 3 \\ -3 & -4 \end{pmatrix}$$

The matrix **M** represents an enlargement with scale factor k and centre (0, 0), where $k > 0$, followed by a rotation anticlockwise through an angle θ about (0, 0).

a. Find the value of k **[2]**

b. Find the value of θ, giving your answer in radians to 2 decimal places. **[2]**

c. Find $\mathbf{M}^{-1}$ **[2]**

(Total 10 marks)

2.

$$\mathrm{f}(z) = z^4 - 6z^3 + 38z^2 - 94z + 221$$

a. Given that $z = 2 + 3\mathrm{i}$ is a root of the equation f(z) = 0, use algebra to find the three other roots of f(z) = 0 **[7]**

b. Show the four roots of f(z) = 0, on a single Argand diagram. **[2]**

(Total 9 marks)

3. The hyperbola H has equation

$$xy = 3$$

The point $Q(1, 3)$ is on H.

a. Find the equation of the normal to H at Q in the form $y = ax + b$, where a and b are constants. **[5]**

The normal at Q intersects H again at the point R

b. Find the coordinates of R **[5]**

(Total 10 marks)

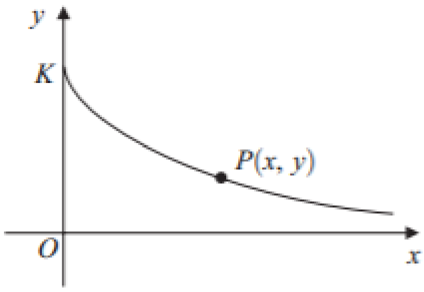
4a. Given that, $y = \operatorname{sech} t$, show that:

i. $\frac{dy}{dt} = -\operatorname{sech} t \tanh t$ **[3]**

ii. $\left(\frac{dy}{dt}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$ **[2]**

b. The diagram shows a sketch of part of the curve given parametrically by,

$$x = t - \tanh t$$
$$y = \operatorname{sech}^4 t$$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve. The arc length KP is denoted by s. Show that:

i. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tanh^2 t$ **[4]**

ii. $s = \ln \cosh t$ **[3]**

iii. $y = e^{-s}$ **[2]**

c. The arc KP is rotated through 2π radians about the x-axis. Show that the surface area generated is,

$$2\pi(1 - e^{-s})$$

[4]

(Total 18 marks)

5. Prove by induction that for all integers $n \geq 1$

$$\sum_{r=1}^{n} (r^2 + 1)(r!) = n(n + 1)!$$

[7]

(Total 7 marks)

6. Given that,

$$\left(z^4 - e^{i\theta}\right)\left(z^4 - e^{-i\theta}\right) = z^8 - 2z^4 \cos\theta + 1$$

Solve the equation,

$$z^8 - z^4 + 1 = 0$$

Giving your answer sin the form $e^{i\emptyset}, where - \pi < \emptyset \leq \pi$ **[6]**

(Total 6 marks)

End of Paper

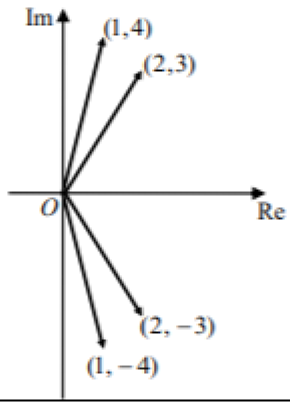
Total Marks: 60

Mark Scheme

Question 1

Question Number	Scheme	Notes	Marks
7. (i)(a)	Reflection	Reflection	B1
	in the y-axis.	**dependent on the previous B mark** Allow y-axis **or** $x = 0$	dB1
			(2)
(i)(a) **Way 2**	Stretch scale factor -1	Stretch scale factor -1	B1
	parallel to the x-axis	**dependent on the previous B mark** parallel to the x-axis	dB1
			(2)
(b)	$\{\mathbf{B} =\}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & ... \\ ... & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & ... \\ ... & 3 \end{pmatrix}$	M1
		Correct matrix	A1
			(2)
	Note: Parts (ii)(a) and (ii)(b) can be marked together.		
(ii)(a)	$\{k =\}\ \sqrt{(-4)^2 - (3)(-3)};= 5$ **or** $k\cos\theta = -4,\ k\sin\theta = -3$ to give $\theta = ...$ and then $k = ...$	Attempts $\sqrt{\pm 16 \pm 9}$ **or** uses full method of trigonometry to find $k = ...$	M1;
		5 only	A1 **cao**
			(2)
(b)	$5\cos\theta = -4,\ 5\sin\theta = -3,\ \tan\theta = \frac{3}{4}$ or $\tan^{-1}\left(\frac{3}{4}\right)$ and e.g. $\theta = \pi + \tan^{-1}\left(\frac{3}{4}\right)$	Uses trigonometry to find an expression in the range $(3.14..., 4.71...)$ or $(-3.14..., -1.57...)$ or $(180°, 270°)$ or $(-180°, -90°)$	M1
	$\{\theta = \pi + 0.64350...\} = 3.78509...\ \{= 3.79\ (2\text{ dp})\}$	awrt 3.79 or awrt -2.50	A1
			(2)
(c)	$\{\mathbf{M}^{-1} =\}\ \frac{1}{25}\begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$	$\frac{1}{25}$ or $\begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$	M1
		$\frac{1}{25}\begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$ or $\begin{pmatrix} -0.16 & -0.12 \\ 0.12 & -0.16 \end{pmatrix}$ o.e.	A1 o.e.
			(2)
			10

Question 2

Question Number	Scheme	Notes	Marks
2.	$f(z) = z^4 - 6z^3 + 38z^2 - 94z + 221,\ z_1 = 2 + 3\text{i}$ satisfies $f(z) = 0$		
(a)	$\{z_2 =\}\ 2 - 3\text{i}$	$2 - 3\text{i}$ seen or used in part (a)	B1
	$z^2 - 4z + 13$	Attempt to expand $(z-(2+3\text{i}))(z-(2-3\text{i}))$ or $(z-(2+3\text{i}))(z-(\text{their complex } z_2))$ or any valid method ***to establish a quadratic factor*** e.g. $z = 2 \pm 3\text{i} \Rightarrow z - 2 = \pm 3\text{i} \Rightarrow z^2 - 4z + 4 = -9$ or sum of roots $= 4$, product of roots 13 to give $z^2 \pm (\text{their sum})z + (\text{their product})$	M1
		$z^2 - 4z + 13$	A1
	$(z^2 - 4z + 13)(z^2 - 2z + 17)$	Attempts to find the other quadratic factor. e.g. using long division to obtain either $z^2 \pm kz + ...,\ k = \text{value} \neq 0$ or $z^2 \pm \alpha z + \beta,\ \beta = \text{value} \neq 0,\ \alpha$ can be 0 or e.g. factorising to obtain either $f(z) = (z^2 - 2z + 5)(z^2 \pm kz \pm c),\ k = \text{value} \neq 0$ or $f(z) = (z^2 - 2z + 5)(z^2 \pm \alpha z \pm \beta),\ \beta = \text{value} \neq 0,\ \alpha$ can be 0	M1
		$z^2 - 2z + 17$	A1
	$\{z^2 - 2z + 17 = 0 \Rightarrow\}$		
	Either • $z = \dfrac{--2 \pm \sqrt{(-2)^2 - 4(1)(17)}}{2(1)}$ • $(z-1)^2 - 1 + 17 = 0 \Rightarrow z = ...$	**dependent on only the previous M mark** Correct method of applying the quadratic formula or completing the square for solving a 3TQ on their 2nd quadratic factor	dM1
	$\{z =\}\ 1 + 4\text{i},\ 1 - 4\text{i}$	$1 + 4\text{i}$ **and** $1 - 4\text{i}$	A1
			(7)
(b)		<u>**Criteria**</u> • $2 \pm 3\text{i}$ plotted correctly in quadrants 1 and 4 • **Dependent on the final M mark being awarded in part (a).** Their final two roots are plotted correctly	
		Satisfies at least one of the criteria	B1ft
		Satisfies both criteria with some indication of scale or coordinates stated with at least one pair of roots symmetrical about the real axis	B1ft
			(2)
			9

Question 3

Question Number	Scheme	Marks
4(a)	$xy = 3$ or $y = \frac{3}{x}$	
	$x\frac{dy}{dx} + y = 0$	
	$\frac{dy}{dx} = \frac{-y}{x}$ or $\frac{dy}{dx} = -\frac{3}{x^2}$	M1A1
	Gradient of normal is $\frac{x}{y}$ or $\frac{x^2}{3}$	M1
	$y - 3 = \frac{1}{3}(x - 1)$	M1
	$y = \frac{1}{3}x + \frac{8}{3}$	A1
		(5)
4(b)	At R, $y = \frac{3}{x}$	M1
	$\frac{9}{x} - x = 8$	
	$x^2 + 8x - 9 = 0$	A1
	$(x+9)(x-1) = 0$	M1
	$x = -9$, $y = -\frac{1}{3}$	A1,A1
		(5) **[10]**

Question 4

4(a)(i)	$\frac{d}{dt}\left(\frac{1}{\cosh t}\right)=-1(\cosh t)^{-2}\sinh t$	M1A1		Or $\frac{-2\left(e^{t}-e^{-t}\right)}{\left(e^{t}+e^{-t}\right)^{2}}$
	$=-\operatorname{sech} t \tanh t$	A1	3	AG
(ii)	Use of $\tanh^2 t = 1-\operatorname{sech}^2 t$	M1		
	Printed result	A1	2	
(b)(i)	$\dot{x}=1-\operatorname{sech}^2 t \quad (\dot{y}=-\operatorname{sech} t \tanh t)$	B1		
	$\dot{x}^2+\dot{y}^2=\left(1-\operatorname{sech}^2 t\right)^2+\operatorname{sech}^2 t-\operatorname{sech}^4 t$	M1A1		Any form
	$=1-\operatorname{sech}^2 t=\tanh^2 t$	A1	4	AG
(ii)	$s=\int_0^t \tanh t \, dt$	M1		Ignore limits for M1 and first A1
	$=[\ln\cosh t]_0^t$	A1		
	$=\ln\cosh t$	A1	3	AG
(iii)	$e^s=\cosh t$	M1		
	$y=e^{-s}$	A1	2	AG
(c)	$S=2\pi\int_0^t \operatorname{sech} t \tanh t \, dt$	M1		Ignore limits for M1 and first A1
	$=2\pi[-\operatorname{sech} t]_0^t$	A1		
	$=2\pi(1-\operatorname{sech} t)$	A1		
	$=2\pi\left(1-e^{-s}\right)$	A1	4	AG
	Total		**18**	

Question 5

Q	Solution	Marks	Total	Comments
5	Assume result true for $n=k$			
	Then $\sum_{r=1}^{k+1}(r^2+1)r!$			
	$=((k+1)^2+1)(k+1)!+k(k+1)!$	M1A1		
	Taking out $(k+1)!$ as factor	m1		
	$=(k+1)!(k^2+2k+1+1+k)$	A1		
	$=(k+1)(k+2)!$	A1		
	$k=1$ shown $(1^2+1)1!=2$, $1\times 2!=2$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 true	E1	7	If all 6 marks earned
	Total		7	

Question 6

(b)	$2\cos\theta = 1$	M1		**SC** If 'hence' not used and, say, $z^8 - z^4 + 1 = 0$ is solved by formula, lose M1A1, but then continue M1m1 etc if $\frac{\pi}{3}$ is obtained
	$\theta = \frac{\pi}{3}$	A1		
	$z^4 = e^{\frac{\pi i}{3}}$ or $e^{\frac{-\pi i}{3}}$	M1		
	$z = e^{\frac{\pi i}{12} + \frac{2k\pi i}{4}}$ or $e^{\frac{-\pi i}{12} + \frac{2k\pi i}{4}}$	m1		
	$e^{\pm\frac{\pi i}{12}}, e^{\pm\frac{7\pi i}{12}}, e^{\pm\frac{5\pi i}{12}}, e^{\pm\frac{11\pi i}{12}}$	A2, 1, 0F	6	A1 if 3 roots correct