A-level Further Mathematics Further Pure 1 and 2 Paper I

1.

$$z = 3 + 2i, \qquad w = 1 - i$$

Find in the form, $a + bi$, where a and b are real constants,

a. zw **[2]**

b. $\frac{z}{w^*}$, showing clearly how you obtained your answer. **[3]**

Given that,

$$|z + k| = \sqrt{53}, \text{ where } k \text{ is a real constant.}$$

c. Find the possible values of k **[4]**

(Total 9 marks)

2. The cubic equation

$$x^3 + px^2 + qx + r = 0$$

Where p, q and r are real, has roots α, β and γ

a. Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

Find the values of p and q **[5]**

b. Given further that one root is 3 + i, find the value of r **[5]**

(Total 10 marks)

3a. Prove by induction that,

$$2 + (3 \times 2) + (4 \times 2^2) + \cdots + (n+1)2^{n-1} = n2^n$$

For all integers $n \geq 1$ **[6]**

b. Show that,

$$\sum_{r=n+1}^{2n} (r+1)2^{(r-1)} = n2^n(2^{n+1} - 1)$$

[3]

(Total 9 marks)

4.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

a. Describe fully the single geometrical transformation represented by the matrix **A** **[3]**

b. Hence find the smallest positive integer value of n for which,

$$\boldsymbol{A}^n = \mathbf{I}$$

Where **I** is the 2 x 2 identity matrix. **[1]**

The transformation represented by the matrix **A** followed by the transformation represented by the matrix **B** is equivalent to the transformation represented by the matrix **C**.

Given that $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ -3 & 5 \end{pmatrix}$

c. Find the matrix **B** **[4]**

(Total 8 marks)

5. Solve the equation

$$7\operatorname{sech} x - \tanh x = 5$$

Give your answer in the form ln a where a is a rational number. **[5]**

(Total 5 marks)

6.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

a. Show that 7 is an eigenvalue of the matrix **M** and find the other two eigenvalues of **M** **[5]**

b. Find an eigenvector corresponding to the eigenvalue 7. **[4]**

(Total 9 marks)

7. The plane P had equation,

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

a. Find a vector perpendicular to the plane P **[2]**

The line l passes through the point $A(1, 3, 3)$ and meets P at (3, 1, 2).

The acute angle between the plane P and the line l is α

b. Find α to the nearest degree. **[4]**

c. Find the perpendicular distance from A to the plane P **[4]**

(Total 10 marks)

End of Paper

Total Marks: 60

Mark Scheme

Question 1

Question Number	Scheme	Notes	Marks
1. (a)	$\{(3+2\text{i})(1-\text{i})\} = 3-3\text{i}+2\text{i}+2$	At least 3 correct terms	M1
	$= 5-\text{i}$	**cao** (Correct answer **only** scores both marks)	A1
			(2)
(b)	$w^* = 1+\text{i}$	Understanding that $w^* = 1+\text{i}$	B1
	$\left\{\frac{z}{w^*} =\right\} \frac{3+2\text{i}}{1+\text{i}} \times \frac{1-\text{i}}{1-\text{i}}$	Multiplies top and bottom by the conjugate of the denominator	M1
	$\left\{= \frac{3-3\text{i}+2\text{i}+2}{1+1}\right\} = \frac{5}{2} - \frac{1}{2}\text{i}$	$\frac{5}{2} - \frac{1}{2}\text{i}$ or $2.5-0.5\text{i}$	A1
			(3)
(c)	$\left\{\lvert 3+2\text{i}+k \rvert = \sqrt{53} \Rightarrow\right\} (3+k)^2 + 4; = 53$	Substitutes for z and uses Pythagoras correctly.	M1;
		Correct equation in any form	A1
	$(3+k)^2 + 4 = 53 \Rightarrow (3+k)^2 = 49 \Rightarrow k =$ or $(3+k)^2 + 4 = 53 \Rightarrow k^2 + 6k - 40 = 0$ $\Rightarrow (k-4)(k+10) = 0 \Rightarrow k =$	**dependent on the previous M mark** Attempt to solve for k	dM1
	$\{k =\}$ 4, -10	Both $\{k =\}$ 4 , -10	A1
			(4)
			9

Question 2

2(a)	$p=-4$	B1		
	$(\alpha+\beta+\gamma)^2=\sum\alpha^2+2\sum\alpha\beta$	M1		
	$16=20+2\sum\alpha\beta$	A1		
	$\sum\alpha\beta=-2$	A1F		
	$q=-2$	A1F	5	
(b)	$3-\mathrm{i}$ is a root	B1		
	Third root is -2	B1F		
	$\alpha\beta\gamma=(3+\mathrm{i})(3-\mathrm{i})(-2)$	M1		
	$=-20$	A1F		Real $\alpha\beta\gamma$
	$r=+20$	A1F	5	Real r

Question 3

Q	Solution	Marks	Total	Comments
4(a)	Assume result true for $n = k$			
	$\sum_{r=1}^{k}(r+1)2^{r-1} = k2^k$			
	$\sum_{r=1}^{k+1}(r+1)2^{r-1} = k2^k + (k+2)2^k$	M1A1		
	$= 2^k(k+k+2)$	m1		
	$= 2^k(2k+2)$			
	$= 2^{k+1}(k+1)$	A1		
	$n=1 \quad 2\times 2^0 = 2 = 1\times 2^1$	B1		
	$P_k \Rightarrow P_{k+1}$ and P_1 is true	E1	6	Provided previous 5 marks earned
(b)	$\sum_{r=1}^{2n}(r+1)2^{r-1} - \sum_{r=1}^{n}(r+1)2^{r-1}$	M1		Sensible attempt at the difference between 2 series
	$= 2n\ 2^{2n} - n2^n$	A1		
	$= n\left(2^{n+1}-1\right)2^n$	A1	3	AG
	Total		**9**	

Question 4

Question Number	Scheme	Notes	Marks
4. (a)	Rotation	Rotation	B1
	225 degrees (anticlockwise)	225 degrees or $\frac{5\pi}{4}$ (anticlockwise) or 135 degrees clockwise	B1 o.e.
	about (0, 0)	**This mark is dependent on at least one of the previous B marks being awarded.** About (0, 0) or about O or about the origin	dB1
	Note: Give 2nd B0 for 225 degrees clockwise		**(3)**
(b)	$\{n=\}\ 8$	8	B1 **cao**
			(1)
(c) **Way 1**	$\mathbf{A}^{-1}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ or $\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$	Correct matrix	B1
	$\{\mathbf{B}=\mathbf{CA}^{-1}\}=\begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}=\ldots$	Attempts $\mathbf{CA}^{-1}$ and finds at least one element of the matrix **B**	M1
	$=\begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ -\sqrt{2} & 4\sqrt{2} \end{pmatrix}$	**dependent on the previous B1M1 marks** At least 2 correct elements	A1
		All elements are correct	A1
			(4)
(c) **Way 2**	$\{\mathbf{BA}=\}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}=\begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix}$	Correct statement using 2×2 matrices. All 3 matrices must contain four elements. **(Can be implied).** (Allow one slip in copying down **C**)	B1
	$-\frac{a}{\sqrt{2}}-\frac{b}{\sqrt{2}}=2,\ \frac{a}{\sqrt{2}}-\frac{b}{\sqrt{2}}=4$ or $-\frac{c}{\sqrt{2}}-\frac{d}{\sqrt{2}}=-3,\ \frac{c}{\sqrt{2}}-\frac{d}{\sqrt{2}}=-5$ and finds at least one of either a or b or c or d	Applies $\mathbf{BA}=\mathbf{C}$ and attempts simultaneous equations in a and b **or** c and d **and** finds at least one of either a or b or c or d	M1
	$=\begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ -\sqrt{2} & 4\sqrt{2} \end{pmatrix}$ or $a=\sqrt{2}, b=-3\sqrt{2}, c=-\sqrt{2}, d=4\sqrt{2}$	**dependent on the previous B1M1 marks** At least 2 correct elements	A1
		All elements are correct	A1
			(4)
			8

Question 5

Question Number	Scheme	Marks
Q1	$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5 \Rightarrow \frac{14}{e^x + e^{-x}} - \frac{(e^x - e^{-x})}{e^x + e^{-x}} = 5$	M1
	$\therefore 14 - (e^x - e^{-x}) = 5(e^x + e^{-x}) \Rightarrow 6e^x - 14 + 4e^{-x} = 0$	A1
	$\therefore 3e^{2x} - 7e^x + 2 = 0 \Rightarrow (3e^x - 1)(e^x - 2) = 0$	M1
	$\therefore e^x = \frac{1}{3}$ or 2	A1
	$x = \ln(\frac{1}{3})$ or $\ln 2$	B1ft **[5]**

Question 6

Question Number	Scheme	Marks
Q3 (a)	$\begin{vmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \therefore (6-\lambda)(7-\lambda)(2-\lambda)+3(7-\lambda)=0$	M1
	$(7-\lambda)=0$ verifies $\lambda = 7$ is an eigenvalue (can be seen anywhere)	M1
	$\therefore (7-\lambda)\{12-8\lambda+\lambda^2+3\}=0 \quad \therefore (7-\lambda)\{\lambda^2-8\lambda+15\}=0$	A1
	$\therefore (7-\lambda)(\lambda-5)(\lambda-3)=0$ and 3 and 5 are the other two eigenvalues	M1 A1 (5)
(b)	Set $\begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	M1
	Solve $-x+y-z=0$ and $3x-y-5z=0$ to obtain $x = 3z$ or $y = 4z$ and a second equation which can contain 3 variables	M1 A1
	Obtain eigenvector as $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ (or multiple)	A1 (4) [9]

Question 7

Question Number	Scheme	Marks
6. **(a)**	$\mathbf{n} = (2\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ o.a.e. (e.g. $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$)	M1 A1 (2)
(b)	Line l has direction $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ Angle between line l and normal is given by $(\cos\beta \text{ or } \sin\alpha) = \frac{4+2+2}{\sqrt{9}\sqrt{9}} = \frac{8}{9}$ $\alpha = 90 - \beta = 63$ degrees to nearest degree.	B1 M1 A1ft A1 awrt (4)
(c) Alt 1	Plane P has equation $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 1$ Perpendicular distance is $\frac{1-(-7)}{\sqrt{9}} = \frac{8}{3}$	M1 A1 M1 A1 (4) **10**
(c) Alt 2	Parallel plane through A has equation $\mathbf{r}.\frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{3} = \frac{-7}{3}$ Plane P has equation $\mathbf{r}.\frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{3} = \frac{1}{3}$ So O lies between the two and perpendicular distance is $\frac{1}{3} + \frac{7}{3} = \frac{8}{3}$	M1 A1 M1 A1 (4)
(c) Alt 3	Distance A to (3,1,2) = $\sqrt{2^2 + 2^2 + 1^2} = 3$ Perpendicular distance is '3' sin $\alpha = 3 \times \frac{8}{9} = \frac{8}{3}$	M1A1 M1A1 (4)
(c) Alt 4	Finding Cartesian equation of plane P: $2x - y - 2z - 1 = 0$ $d = \frac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d \rvert}{\sqrt{n_1^2 + n_2^2 + n_3^2}} = \frac{\lvert 2(1) - 1(3) - 2(3) - 1 \rvert}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{8}{3}$	M1 A1 M1A1 (4)