A-level Further Mathematics Further Pure 1 and 2
Paper H

1a. The parabola C has cartesian equation $y^2 = 12x$

The point $P(3p^2, 6p)$ lies on C, where $p \neq 0$

a. Show that the equation of the normal to the curve C at the point P is,

$$y + px = 6p + 3p^3$$

[5]

This normal crosses the curve C again at the point Q.

Given that $p = 2$ and that S is the focus of the parabola, find

b. The coordinates of the point Q, **[5]**

c. The area of the triangle, PQS **[4]**

(Total 14 marks)

2. The quadratic equation,

$$4x^2 + 3x + 1 = 0$$

has roots, α and β.

a. Write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$ **[2]**

b. Find the value of $(\alpha^2 + \beta^2)$ **[2]**

c. Find a quadratic equation which has roots,

$$(4\alpha - \beta) \text{ and } (4\beta - \alpha)$$

Giving your answer in the form $px^2 + qx + r = 0$ where p, q *and* r are integers to be determined. **[4]**

(Total 8 marks)

3. It is given that, $f(x) = \tanh^{-1}\left(\frac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$

a. Show that $f'(x) = -\frac{1}{1+2x}$, and find $f''(x)$ **[6]**

b. Show that the first three of the Maclaurin series for $f(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found.

[4]

(Total 10 marks)

4.

$$M = \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that, $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$ is an eigen vector of the matrix **M,**

a. Find the eigenvalue of **M** corresponding to $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$ **[2]**

b. Show that $k = -7$ **[2]**

c. Find the other two eigenvalues of the matrix **M** **[4]**

The image of the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ under the transformation represented by **M** is $\begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$.

d. Find the values of the constants, p, q and r. **[4]**

(Total 12 marks)

5. The curve C has parametric equations,

$$x = 3t^4, \qquad y = 4t^3, \qquad 0 \leq t \leq 1$$

The curve is rotated through 2π radians about the x-axis. The area of the curved surface generated is S.

a. Show that,

$$S = k\pi \int_0^1 t^5 (t^2 + 1)^{\frac{1}{2}} \, dt$$

where k is a constant to be found. **[4]**

b. Use the substitution $u^2 = t^2 + 1$ to find the value of S, giving your answer in the form $p\pi(11\sqrt{2} - 4)$ where p is a rational number to be found.

[7]

(Total 11 marks)

6.Without using a calculator, find,

$$\int_{-2}^{1} \frac{1}{x^2 + 4x + 13} dx,$$

giving your answer as a multiple of π **[5]**

(Total 5 marks)

End of Paper

Total Marks: 60

<u>Mark Scheme</u>

Question 1

Question Number	Scheme	Notes	Marks
4(a)	$y^2=12x \Rightarrow y=\sqrt{12}x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\sqrt{12}x^{-\frac{1}{2}}$	$\frac{dy}{dx}=kx^{-\frac{1}{2}}$	
	$y^2=12x \Rightarrow 2y\frac{dy}{dx}=12$	$\alpha y\frac{dy}{dx}=\beta$	M1
	$\frac{dy}{dx}=\frac{dy}{dp}.\frac{dp}{dx}=6.\frac{1}{6p}$	their $\frac{dy}{dp}\times\left(\frac{1}{\text{their }\frac{dx}{dp}}\right)$	
	$\frac{dy}{dx}=\frac{1}{2}\sqrt{12}x^{-\frac{1}{2}}$ or $2y\frac{dy}{dx}=12$ or $\frac{dy}{dx}=6.\frac{1}{6p}$ or equivalent expressions	Correct differentiation	A1
	$m_T=\frac{1}{p}\Rightarrow m_N=-p$	Correct perpendicular gradient rule	M1
	$y-6p=-p\left(x-3p^2\right)$	$y-6p=$ their $m_N\left(x-3p^2\right)$ or $y=mx+c$ with their m_N and $(3p^2,6p)$ in an attempt to find 'c'. **Their m_N must have come from calculus and should be a function of p which is not their tangent gradient.**	M1
	$y+px=6p+3p^3$ *	Achieves printed answer with no errors	A1*
			(5)
(b)	$p=2\Rightarrow y+2x=12+24$	Substitutes the given value of p into the normal	M1
	$y+\frac{y^2}{6}=36$	Substitutes to obtain an equation in one variable (x, y or "q")	M1
	$y^2+6y-216=0$		
	$(y+18)(y-12)=0\Rightarrow y=$	Solves their 3TQ	M1
	$y=-18\Rightarrow x=27$	A1: One correct coordinate A1: Both coordinates correct	A1, A1
			(5)
(c)	Focus is (3, 0) or $a=3$ or OS = 3	Must be seen or used in (c)	B1
	$y=0\Rightarrow x=18$		
	$A=\frac{1}{2}(18-3)(12)+\frac{1}{2}(18-3)(18)$	M1: Correct attempt at area A1: Correct expression	M1A1
	A = 225	Correct area	A1
			(4)
			Total 14

Question 2

Question Number	Scheme	Notes	Marks
5(a)	$\alpha+\beta=-\frac{3}{4},\ \ \alpha\beta=\frac{1}{4}$		B1, B1
			(2)
(b)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{9}{16}-\frac{1}{2}=\frac{1}{16}$	M1:Use of $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ A1: $\frac{1}{16}$ cso (allow 0.0625)	M1 A1
			(2)
(c)	Sum $4\alpha-\beta+4\beta-\alpha=3(\alpha+\beta)=-\frac{9}{4}$	Attempt numerical sum	M1
	Product $(4\alpha-\beta)(4\beta-\alpha)=17\alpha\beta-4(\alpha^2+\beta^2)$ $=\frac{17}{4}-\frac{1}{4}=4$	Attempt numerical product	M1
	$x^2-(-\frac{9}{4})x+4(=0)$	Uses x^2 – (sum)x + (prod) with sum, prod numerical (= 0 not reqd.)	M1
	$4x^2+9x+16=0$	Any multiple (including = 0)	A1
			(4)
			Total 8

Question 3

7	(i)	Attempt quotient/product on bracket	M1	
		Get $-3/(2+x)^2$	A1	May be implied
		Use Formulae Booklet or derive from tanh $y = (1-x)/(2+x)$	M1	Attempt $\tanh^{-1}$ part in terms of x
		Get $\frac{-3}{(2+x)^2} \cdot \frac{1}{1-((1-x)/(2+x))^2}$	A1√	From their results above
		Clearly tidy to A.G.	A1	
		Get $f''(x) = 2/(1+2x)^2$	B1	cao
			6	
			SC	Use reasonable ln definition M1 Get $y=½\ln((1-k)/(1+k))$ for $k=(1-x)/(1+2x)$ A1 Tidy to $y=½\ln(3/(1+2x))$ A1 Attempt chain rule M1 Clearly tidy to A.G. A1 Get $f''(x)$ B1
	(ii)	Attempt f(0), f ′(0) and f ″(0)	M1	From their differentiation
		Get $\tanh^{-1}$ ½, -1 and 2	A1√	
		Replace $\tanh^{-1}$ ½ = ½ ln3 (=ln√3)	B1	Only
		Get $\ln\sqrt{3} - x + x^2$	A1	
			4	
			SC	Use standard expansion from ½ln3-½ln(1+2x)

Question 4

Question Number	Scheme	Notes	Marks
3(a)	$\begin{pmatrix}0 & 1 & 9\\1 & 4 & k\\1 & 0 & -3\end{pmatrix}\begin{pmatrix}7\\19\\1\end{pmatrix}=\lambda\begin{pmatrix}7\\19\\1\end{pmatrix}$	Correct statement	M1
	$7-3=\lambda$ or $28=7\lambda \Rightarrow \lambda=4$	Correct eigenvalue	A1
			(2)
(b)	$7+4\times19+k=4\times19 \Rightarrow k=-7$ *	M1: Uses y component to establish an equation for k A1*: Correct k	M1A1*
			(2)
(c)	$\begin{vmatrix}0-\lambda & 1 & 9\\1 & 4-\lambda & -7\\1 & 0 & -3-\lambda\end{vmatrix}=0$		
	$\lambda(4-\lambda)(3+\lambda)+(3+\lambda)-7+9(\lambda-4)=0$ or $-7+9(\lambda-4)-(3+\lambda)\left[\lambda(\lambda-4)-1\right]$	M1: Correct characteristic equation method (allow sign errors only) A1: Correct equation in any form	M1A1
	$(4-\lambda)\left[\lambda(3+\lambda)-1-9\right]=0$	NB $\lambda^3-\lambda^2-22\lambda+40=0$	
	$(\lambda-2)(\lambda+5)=0 \Rightarrow \lambda=2,-5$	A1: $\lambda=2$ or $\lambda=-5$ A1: $\lambda=2$ and $\lambda=-5$	A1A1
			(4)
(d) Way 1	$\begin{pmatrix}0 & 1 & 9\\1 & 4 & -7\\1 & 0 & -3\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}q+9r\\p+4q-7r\\p-3r\end{pmatrix}$	Multiplies by **M** to obtain a vector in terms of p, q and r	M1
	$\begin{pmatrix}q+9r\\p+4q-7r\\p-3r\end{pmatrix}=\begin{pmatrix}-6\\21\\5\end{pmatrix}$	Correct equations	A1
	$p=2, q=3, r=-1$	M1: Solves simultaneously to obtain at least one of p, q or r. **Dependent on the previous method mark.** A1: Correct answers	dM1A1
	Correct equations followed by correct answers scores full marks in part (d)		
			(4)
(d) Way 2	$\begin{pmatrix}p\\q\\r\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}-6\\21\\5\end{pmatrix}=\frac{1}{40}\begin{pmatrix}12 & -3 & 43\\4 & 9 & -9\\4 & -1 & 1\end{pmatrix}\begin{pmatrix}-6\\21\\5\end{pmatrix}$	M1: An appreciation that $p\mathbf{i}+q\mathbf{j}+r\mathbf{k}=\mathbf{M}^{-1}(-6\mathbf{i}+21\mathbf{j}+5\mathbf{k})$ A1: Correct inverse	M1A1
	$\begin{pmatrix}p\\q\\r\end{pmatrix}=\frac{1}{40}\begin{pmatrix}12 & -3 & 43\\4 & 9 & -9\\4 & -1 & 1\end{pmatrix}\begin{pmatrix}-6\\21\\5\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$	M1: Multiplies their inverse by the given vector. **Dependent on the previous method mark.** A1: Correct vector	dM1A1
			(4)

Question 5

Question Number	Scheme	Notes	Marks
7	$x=3t^4,\ \ y=4t^3$		
(a)	$\frac{dx}{dt}=12t^3,\ \frac{dy}{dt}=12t^2$	Correct derivatives	B1
	$S=(2\pi)\int y\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)^{\frac{1}{2}}dt=(2\pi)\int 4t^3\sqrt{(12t^3)^2+(12t^2)^2}\,dt$ $\left(=(2\pi)\int 4t^3(144t^6+144t^4)^{\frac{1}{2}}dt\right)$ M1: Substitutes their derivatives into a correct formula (2π not required)		M1
	$S=(2\pi)\int 4t^3(144t^4)^{\frac{1}{2}}(t^2+1)^{\frac{1}{2}}dt$	Attempt to factor out at least t^4 - numerical factor may be left	M1
	$S=96\pi\int_0^1 t^5(t^2+1)^{\frac{1}{2}}dt$	Correct completion	A1
			(4)
(b)	$u^2=t^2+1\Rightarrow 2u\frac{du}{dt}=2t$ or $2u=2t\frac{dt}{du}$	Correct differentiation	B1
	$t=0\Rightarrow u=1,\ t=1\Rightarrow u=\sqrt{2}$	Correct limits **ALT:** reverse the substitution later. (Treat as M1 in this case and award later when work seen)	B1
	$S=(96\pi)\int t^5\times u\times\frac{u}{t}du$		
	$S=(96\pi)\int (u^2-1)^2\times u^2\,du$	M1: Complete substitution A1: Correct integral in terms of u. Ignore limits, need not be simplified	M1A1
	$S=(96\pi)\int (u^6-2u^4+u^2)du=(96\pi)\left[\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}\right]$ M1: Expands and attempts to integrate		dM1
	$S=96\pi\left[\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}\right]_1^{\sqrt{2}}=96\pi\left\{\left(\frac{\sqrt{2}^7}{7}-\frac{2\sqrt{2}^5}{5}+\frac{\sqrt{2}^3}{3}\right)-\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right)\right\}$ M1: Correct use of their changed limits (both to be changed) **ALT:** If sub reversed, substitute the original limits		ddM1
	$S=\frac{192\pi}{105}(11\sqrt{2}-4)$	Cao eg $\frac{64\pi}{35}$	A1
			(7)
			Total 11

Question 6

Question Number	Scheme	Notes	Marks
3(a)	$x^2+4x+13 \equiv (x+2)^2+9$		B1
	$\int \frac{1}{(x+2)^2+9}\,dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)$	M1: $k\arctan f(x)$. A1: Correct expression	M1A1
	$\left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1} = \frac{1}{3}(\arctan 1 - \arctan 0)$	Correct use of limits arctan0 need not be shown	M1
	$\frac{\pi}{12}$	cao	A1
			(5)
ALT:	**Sub** $x+2=3\tan t$		
	$x^2+4x+13 \equiv (x+2)^2+9$		B1
	$\frac{dx}{dt}=3\sec^2 t \qquad x=-2, \tan t=0, t=0;\ x=1, \tan t=1, t=\frac{\pi}{4}$		
	$\int \frac{3\sec^2 t}{9\tan^2 t+9}\,dt = \frac{1}{3}\int dt = \frac{1}{3}t$	M1 sub and integrate inc use of $\tan^2+1=\sec^2$ A1 Correct expression Ignore limits	M1A1
	$..\left[\frac{\pi}{12}\right]_0^{\frac{\pi}{4}}.$	Either change limits and substitute Or reverse substitution and substitute original imits	M1
	$\frac{\pi}{12}$	cao	A1