A-level Further Mathematics Further Pure 1 and 2 Paper G

1a. Show, using the formulae for $\sum r$ and $\sum r^2$, that

$$\sum_{r=1}^{n}(6r^2+4r-1)=n(n+2)(2n+1)$$

[5]

b. Hence, or otherwise, find the value of,

$$\sum_{r=11}^{20}(6r^2+4r-1)$$

[2]

(Total 7 marks)

2. The complex numbers z_1 and z_2 are given by,

$$z_1 = 2 + 8\mathrm{i}$$
$$z_2 = 1 - \mathrm{i}$$

Find, showing your working,

a. $\left|\frac{z_1}{z_2}\right|$ in the form $a+b\mathrm{i}$, where a and b are real. **[3]**

b. The value of, $\left|\frac{z_1}{z_2}\right|$, **[2]**

c. The value of $\arg\frac{z_1}{z_2}$, giving your answer in radians to 2 decimal places. **[2]**

(Total 7 marks)

3. Use the definitions of hyperbolic functions in terms of exponentials to prove that,

$$8\sinh^4 x \equiv \cosh 4x - 4\cosh 2x + 3$$

b. Solve the equation, **[4]**

$$\cosh 4x - 3\cosh 2x + 1 = 0$$

Giving your answer(s) in logarithmic form.

[5]

(Total 9 marks)

4. The hyperbola H has foci at (5, 0) and (-5, 0) and directrices with equations,

$$x = -\frac{9}{5} \text{ and } x = \frac{9}{5}$$

Find a cartesian equation for H. **[7]**

(Total 7 marks)

5a. Prove by induction that, for $n \in Z^+$

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

[6]

b. A sequence of positive integers is defined by,

$$u_1 = 1$$
$$u_{n+1} = u_1 + n(3n+1), \qquad n \in Z^+$$

Prove by induction that,

$$u_n = n^2(n-1) + 1, \qquad n \in Z^+$$

[5]

(Total 11 marks)

6a. Given that $y = sinh^{-1}\, x$, prove that $y = ln\,(x + \sqrt{x^2+1})$ **[3]**

b. It is given that x satisfies the equation $\sinh^{-1} x - \cosh^{-1} x = \ln 2$. Use the logarithmic forms for $\cosh^{-1} x$ to show that,

$$\sqrt{x^2+1} - 2\sqrt{x^2-1} = x$$

Hence, by squaring this equation, find the exact value of x **[5]**

(Total 8 marks)

7. The equation of a curve, in polar coordinates is,

$$r = \sqrt{3} + tan\,\theta, \qquad \text{for } -\tfrac{1}{3}\pi \le \theta \le \tfrac{1}{4}\pi$$

a. Find the equation of the tangent at the pole **[2]**

b. State the greatest value of r and the corresponding value of θ **[2]**

c. Sketch the curve **[2]**

d. Find the exact area of the region enclosed by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{4}\pi$ **[5]**

(Total 11 marks)

End of Paper

Total Marks: 60

Mark Scheme

Question 1

Question Number	Scheme	Marks
2 (a)	$6\sum r^2 + 4\sum r - \sum 1 = 6\frac{n}{6}(n+1)(2n+1) + 4\frac{n}{2}(n+1), -n$	M1 A1, B1
	$= \frac{n}{6}(12n^2 + 18n + 6 + 12n + 12 - 6)$ or $n(n+1)(2n+1) + (2n+1)n$	M1
	$= \frac{n}{6}(12n^2 + 30n + 12) = n(2n^2 + 5n + 2) = n(n+2)(2n+1)$ *	A1 (5)
(b)	$\sum_{r=1}^{20}(6r^2 + 4r - 1) - \sum_{r=1}^{10}(6r^2 + 4r - 1) = 20 \times 22 \times 41 - 10 \times 12 \times 21$	M1
	$= 15520$	A1 (2) [7]

Question 2

Question Number	Scheme	Marks
Q1	(a) $\frac{z_1}{z_2}=\frac{2+8\text{i}}{1-\text{i}}\times\frac{1+\text{i}}{1+\text{i}}$	M1
	$=\frac{2+2\text{i}+8\text{i}-8}{2}=-3+5\text{i}$	A1 A1 (3)
	(b) $\left\lvert\frac{z_1}{z_2}\right\rvert=\sqrt{(-3)^2+5^2}=\sqrt{34}$ (or awrt 5.83)	M1 A1ft (2)
	(c) $\tan\alpha=-\frac{5}{3}$ or $\frac{5}{3}$	M1
	$\arg\frac{z_1}{z_2}=\pi-1.03...=2.11$	A1 (2) [7]
	Notes (a) $\times\frac{1+\text{i}}{1+\text{i}}$ and attempt to multiply out for M1 -3 for first A1, +5i for second A1 (b) Square root required without i for M1 $\frac{\lvert z_1\rvert}{\lvert z_2\rvert}$ award M1 for attempt at Pythagoras for both numerator and denominator (c) tan or $\tan^{-1}$, $\pm\frac{5}{3}$ or $\pm\frac{3}{5}$ seen with their 3 and 5 award M1 2.11 correct answer only award A1	

Question 3

4 (i) $8\sinh^4 x \equiv \frac{8}{16}\left(e^x - e^{-x}\right)^4$	B1	$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ seen or implied
$\equiv \frac{8}{16}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right)$	M1	For attempt to expand $(\ldots)^4$ by binomial theorem *OR* otherwise
$\equiv \frac{1}{2}\left(e^{4x} + e^{-4x}\right) - \frac{4}{2}\left(e^{2x} + e^{-2x}\right) + \frac{6}{2}$	M1	For grouping terms for $\cosh 4x$ *or* $\cosh 2x$ *OR* using e^{4x} *or* e^{2x} expressions from RHS
$\equiv \cosh 4x - 4\cosh 2x + 3$	A1 **4**	For correct expression **AG**
SR may be done wholly from RHS to LHS	M1 M1 B1 A1	Evidence of factorising required for 2nd M1
(ii) METHOD 1 $\cosh 4x - 3\cosh 2x + 1 = 0$		
$\Rightarrow (8\sinh^4 x + 4\cosh 2x - 3) - 3\cosh 2x + 1 = 0$	M1	For using **(i)** and $\cosh 2x \equiv \pm 1 \pm 2\sinh^2 x$
$\Rightarrow 8\sinh^4 x + 2\sinh^2 x - 1 = 0$	A1	For correct equation
$\Rightarrow \left(4\sinh^2 x - 1\right)\left(2\sinh^2 x + 1\right) = 0 \Rightarrow \sinh x = \pm\frac{1}{2}$	M1 A1	For solving their quartic for $\sinh x$ For correct $\sinh x$ (ignore other roots)
$\Rightarrow x = \ln\left(\pm\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	A1√ **5**	For correct answers (and no more) f.t. from their value(s) for $\sinh x$
SR Similar scheme for $8\cosh^4 x - 14\cosh^2 x + 5 = 0 \Rightarrow \cosh x = \frac{1}{2}\sqrt{5} \Rightarrow x = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$		
METHOD 2 $\cosh 4x - 3\cosh 2x + 1 = 0$		
$\Rightarrow (2\cosh^2 2x - 1) - 3\cosh 2x + 1 = 0$	M1	For using $\cosh 4x \equiv \pm 2\cosh^2 2x \pm 1$
$\Rightarrow 2\cosh^2 2x - 3\cosh 2x = 0$	A1	For correct equation
$\Rightarrow \cosh 2x = \frac{3}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	M1 A1	For solving for $\cosh 2x$ For correct $\cosh 2x$ (ignore others)
$= \pm\frac{1}{2}\ln\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)$	A1√	For correct answers (and no more) f.t. from value(s) for $\cosh 2x$
METHOD 3 Put all into exponentials	M1	For changing to $e^{\pm kx}$
$\Rightarrow e^{4x} - 3e^{2x} + 2 - 3e^{-2x} + e^{-4x} = 0$	A1	For correct equation
$\Rightarrow \left(e^{4x} + 1\right)\left(e^{4x} - 3e^{2x} + 1\right) = 0$	M1 A1	For solving for e^{2x} For correct e^{2x} (ignore others)
$\Rightarrow e^{2x} = \frac{1}{2}\left(3 \pm \sqrt{5}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	A1√	For correct answers (and no more) f.t. from value(s) for e^{2x}
	9	

Question 4

Question Number	Scheme		Marks
	Foci $(\pm5, 0)$, Directrices $x = \pm\frac{9}{5}$		
1.	$(\pm)ae = (\pm)5$ and $(\pm)\frac{a}{e} = (\pm)\frac{9}{5}$	Correct equations (ignore ±'s)	B1
	so $e = \frac{5}{a} \Rightarrow \frac{a^2}{5} = \frac{9}{5} \Rightarrow a^2 = 9$ or $a = \frac{5}{e} \Rightarrow \frac{5}{e^2} = \frac{9}{5} \Rightarrow e = \frac{5}{3} \Rightarrow a = 3$	M1: Solves using an appropriate method to find a^2 or a A1: $a^2 = 9$ or $a = (\pm)3$	M1A1
	$b^2 = a^2e^2 - a^2 \Rightarrow b^2 = 25 - 9$ so $b^2 = 16 \quad (\Rightarrow b = 4)$ or $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(\frac{25}{9} - 1\right)$ $b^2 = 16 \quad (\Rightarrow b = 4)$	M1: Use of $b^2 = a^2(e^2 - 1)$ to obtain a numerical value for b^2 or b A1: : $b^2 = 16$ or $b = (\pm)4$	M1 A1
	So $\frac{x^2}{9} - \frac{y^2}{16} = 1$	M1:Use of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with their a^2 and b^2 A1: Correct hyperbola in any form.	M1 A1
			(7)

Question 5

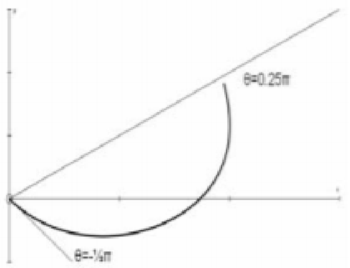
Question Number	Scheme	Marks
8.	(a) If $n=1$, $\sum_{r=1}^{n} r(r+3)=1\times4=4$ and $\frac{1}{3}n(n+1)(n+5)=\frac{1}{3}\times1\times2\times6=4$,	B1
	(so true for $n=1$. Assume true for $n=k$)	
	So $\sum_{r=1}^{k+1} r(r+3)=\frac{1}{3}k(k+1)(k+5)+(k+1)(k+4)$	M1
	$=\frac{1}{3}(k+1)[k(k+5)+3(k+4)]=\frac{1}{3}(k+1)[k^2+8k+12]$	A1
	$=\frac{1}{3}(k+1)(k+2)(k+6)$ which implies is true for $n=k+1$	dA1
	As result is true for $n=1$ this implies true for all positive integers and so result is true by induction	dM1A1cso
		(6)
	(b) $u_1=1^2(1-1)+1=1$	B1
	(so true for $n=1$. Assume true for $n=k$)	
	$u_{k+1}=k^2(k-1)+1+k(3k+1)$	M1,
	$=k(k^2-k+3k+1)+1=k(k+1)^2+1$ which implies is true for $n=k+1$	A1
	As result is true for $n=1$ this implies true for all positive integers and so result is true by induction	M1A1cso **(5)**
		[11]

Question 6

7	(i)		$x = \sinh y = \frac{e^y - e^{-y}}{2}$ $\Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2+1}$ reject – sign as $e^y > 0 \Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$	M1 A1 A1 **[3]**	For correct expression for sinh y and attempt to obtain quadratic For correct solution(s) for e^y For justification of + sign to **AG**	
			Alt: $\sinh y + \cosh y = e^y$ $\sinh y = x \Rightarrow \cosh y = \pm\sqrt{x^2+1}$ reject -ve sign as $e^y > 0$ $\Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow y = \ln\left(x+\sqrt{x^2+1}\right)$			

Question			Answer	Marks	Guidance	
7	(ii)		$\ln\left(x+\sqrt{x^2+1}\right) - \ln\left(x+\sqrt{x^2-1}\right) = \ln 2$ $\Rightarrow \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}} = 2$ $\Rightarrow \sqrt{x^2+1} - 2\sqrt{x^2-1} = x$ $\Rightarrow 4x^2 - 3 = 4\sqrt{x^4-1}$ $\Rightarrow 24x^2 = 25 \Rightarrow x = \frac{5}{\sqrt{24}} \left(= \frac{5}{12}\sqrt{6}\right)$	M1 A1 M1 A1 A1 **[5]**	For stating both ln expressions and attempting to exponentiate For correct equation **AG** For attempting to square once For a correct equation with $\sqrt{\ }$ as subject For correct x and no others **isw**	Removing lns is not an attempt to exponentiate

Question 7

7 (i) Attempt to solve $r=0$, $\tan\theta = -\sqrt{3}$	M1 Allow $\pm\sqrt{3}$
Get $\theta = -\frac{1}{3}\pi$ only	A1 Allow -60°
(ii) $r = \sqrt{3} + 1$ when $\theta = \frac{1}{4}\pi$	B1,B1 AEF for r, 45° for θ
(iii)	B1 Correct r at correct end-values of θ; Ignore extra θ used
	
	B1 Correct shape with r not decreasing
(iv) Formula with correct r used	M1 r^2 may be implied
Replace $\tan^2\theta = \sec^2\theta - 1$	B1
Attempt to integrate <u>their</u> expression	M1 Must be 3 different terms leading to any 2 of $a\theta + b\ln(\sec\theta/\cos\theta) + c\tan\theta$
Get $\theta + \sqrt{3}\ln\sec\theta + \frac{1}{2}\tan\theta$	A1 Condone answer x2 if ½ seen elsewhere
Correct limits to $\frac{1}{4}\pi + \sqrt{3}\ln\sqrt{2} + \frac{1}{2}$	A1 cao; AEF