A-level Further Mathematics Further Pure 1 and 2
Paper F

1. i) Given that,

$$\frac{3w+7}{5} = \frac{p-4i}{3-i}$$

Where p is a real constant.

a. Express w in the form $a + b$i, where a and b are real constants.
Give your answer in its simplest form in terms of p. **[5]**

Given that $\arg w = -\frac{\pi}{2}$
b. Find the value of p **[1]**

ii) Given that,

$$(z + 1 - 2\mathrm{i})^* = 4\mathrm{i}z$$

Find z, giving your answer in the form $z = x + \mathrm{i}y$, where x and y are real constants. **[6]**

(Total 12 marks)

2. The points P$(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

a. Show the equation of the chord PQ is,

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}$$

[4]

b. Write down the coordinates of the mid-point of PQ, **[1]**

Given that the gradient, m, of the chord PQ is a constant,

c. Show that the centre of the chord lies on a line,

$$y = -kx$$

Expressing k in terms of m **[5]**

(Total 10 marks)

3. In the interval $13 < x < 14$, the equation,

$$3 + x\sin\left(\frac{x}{4}\right) = 0$$

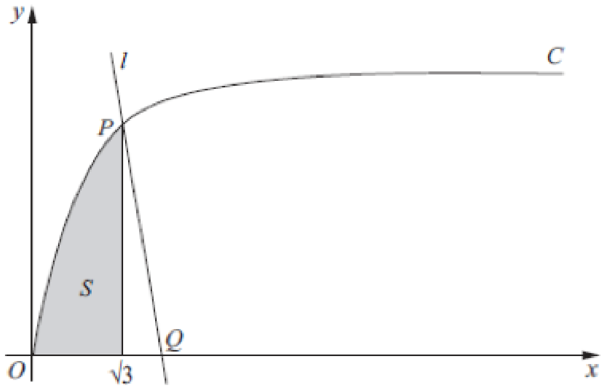
Where x is measured in radians, has exactly one root, α

a. Starting with the interval [13, 14], use interval bisection twice to find an interval of width 0.25 which contains α **[3]**

b. Use linear interpolation once on the interval [13, 14] to find an approximate value for α. Give your answer to 3 decimal places. **[4]**

(Total 7 marks)

4.



The figure shows part of the curve C with parametric equations,

$$x = \tan\theta, \qquad y = \sin\theta, \qquad 0 \leq \theta < \frac{\pi}{2}$$

a. The point P lies on C and has coordinates $(\sqrt{3}, \frac{1}{2}\sqrt{3})$. **[2]**

The line l is a normal to C at P. The normal cuts the x-axis at the point Q.

b. Show that Q had coordinates $(k\sqrt{3}, 0)$, giving the value of the constant k. **[6]**

The finite shaded region S shown in the figure is bounded by the curve C, the line $x = \sqrt{3}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

c. Find the volume of the solid of revolution, giving your answer in the form, $p\pi\sqrt{3} + q\pi^2$, where p and q are constants.

[7]

(Total 15 marks)

5. Find the general solution of the differential equation,

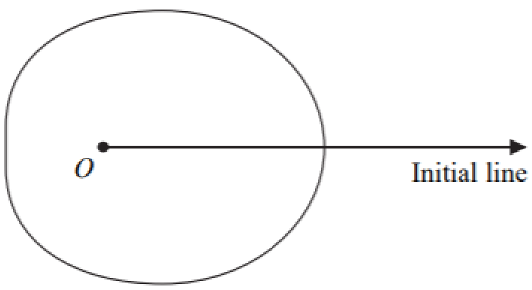
$$sin\, x \frac{dy}{dx} - y\, cos\, x = sin\, 2x\, sin\, x$$

Giving your answer in the form $y = f(x)$ **[8]**

(Total 8 marks)

6. The figure shows a sketch of the curve with polar equation,

$$r = a + 3cos\,\theta, \quad a > 0, \quad 0 \leq \theta < 2\pi$$



The area enclosed by the curve is $\frac{107}{2}\pi$

Find the value of a. **[8]**

(Total 8 marks)

End of Paper

Total Marks: 60

Mark Scheme

Question 1

Question Number	Scheme		Marks
9.(i) (a)	$\dfrac{3w+7}{5} = \dfrac{(p-4\text{i})}{(3-\text{i})} \times \dfrac{(3+\text{i})}{(3+\text{i})}$	Multiplies by $\dfrac{(3+\text{i})}{(3+\text{i})}$ or divide by $(9-3\text{i})$ then multiply by $\dfrac{(9+3\text{i})}{(9+3\text{i})}$	M1
	$= \left(\dfrac{3p+4}{10}\right) + \left(\dfrac{p-12}{10}\right)\text{i}$	Evidence of $(3-\text{i})(3+\text{i}) = 10$ or 3^2+1^2 or 9^2+3^2	B1
	So, $w = \left(\dfrac{3p-10}{6}\right) + \left(\dfrac{p-12}{6}\right)\text{i}$	Rearranges to $w = \dots$	dM1
		At least one of either the real or imaginary part of w is correct in any equivalent form.	A1
		Correct w in the form $a+bi$. Accept $a+ib$.	A1
			[5]
ALT (i) (a)	$(3-\text{i})(3w+7) = 5(p-4\text{i})$		
	$9w+21-3\text{i}w-7\text{i} = 5p-20\text{i}$		
	$w(9-3\text{i}) = 5p-21-13\text{i}$		
	Let $w = a+b\text{i}$, so $(a+b\text{i})(9-3\text{i}) = 5p-21-13\text{i}$		
	$9a+3b-3a\text{i}+9b\text{i} = 5p-21-13\text{i}$		
	Real: $9a+3b = 5p-21$ Imaginary: $-3a+9b = -13$	Sets $w = a+b\text{i}$ and equates at least either the real or imaginary part.	M1
		$9a+3b = 5p-21$	B1
	$b = \dfrac{p-12}{6}$, $a = \dfrac{3p-10}{6}$	Solves to finds $a = \dots$ and $b = \dots$	dM1
		At least one of a or b is correct in any equivalent form.	A1
	$w = \left(\dfrac{3p-10}{6}\right) + \left(\dfrac{p-12}{6}\right)\text{i}$	Correct w in the form $a+bi$. Accept $a+ib$.	A1
			[5]
(b)	$\left\{\arg w = -\dfrac{\pi}{2} \Rightarrow \left(\dfrac{3p-10}{6}\right) = 0\right\} \Rightarrow p = \dfrac{10}{3}$	$p = \dfrac{10}{3}$ Follow through provided $p < 12$	B1ft
			[1]

(ii)	$(x+\text{i}y+1-2\text{i})^* = 4\text{i}(x+\text{i}y)$	Replaces z with $x+\text{i}y$ on both sides of the equation	M1
	$x-\text{i}y+1+2\text{i} = 4\text{i}(x+\text{i}y)$ or $x+\text{i}y+1-2\text{i} = -4\text{i}(x-\text{i}y)$	Fully correct method for applying the conjugate	M1
	$x-\text{i}y+1+2\text{i} = 4\text{i}x-4y$		
	Real: $x+1 = -4y$ Imaginary: $-y+2 = 4x$	$x+1 = -4y$ and $-y+2 = 4x$	A1
	$4x+16y = -4$ $4x+y = 2$ $\Rightarrow 15y = -6 \Rightarrow y = \dots$	Solves two equations in x and y to obtain at least one of x or y	ddM1
	So, $x = \dfrac{3}{5},\ y = -\dfrac{2}{5}$ $\left\{z = \dfrac{3}{5} - \dfrac{2}{5}\text{i}\right\}$	At least one of either x or y are correct	A1
		Both x and y are correct	A1
			[6]
			Total 12

Question 2

<table>
<tr><th>Question</th><th colspan="2">Scheme</th><th>Marks</th></tr>
<tr><td></td><td colspan="2">In this question condone the use of a and/or b for α and β</td><td></td></tr>
<tr><td>6(a)</td><td>Gradient $m = \frac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}$</td><td>Correct attempt at chord gradient – do not allow slips unless a correct method is clear</td><td>M1</td></tr>
<tr><td></td><td>$y - 2\sin\alpha = m(x - 3\cos\alpha)$
or $y - 2\sin\beta = m(x - 3\cos\beta)$
or $y = mx + c$ and attempts to find c using P or Q</td><td>A correct straight line method using their chord gradient and the point P or the point Q</td><td>M1</td></tr>
<tr><td></td><td colspan="2">$y - 2\sin\alpha = \frac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}(x - 3\cos\alpha)$
$y - 2\sin\beta = \frac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}(x - 3\cos\beta)$
$y - 2\sin\alpha = \frac{4\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}{-6\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}(x - 3\cos\alpha)$
$y = \frac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}x + 2\sin\alpha - \frac{3\cos\alpha(2\sin\beta - 2\sin\alpha)}{3\cos\beta - 3\cos\alpha}$
$y = -\frac{2\cos\frac{\alpha+\beta}{2}}{3\sin\frac{\alpha+\beta}{2}}x + 2\sin\alpha + \frac{2\cos\alpha\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$</td><td>A1</td></tr>
<tr><td></td><td colspan="2">A correct equation for the chord in any form.</td><td></td></tr>
<tr><td></td><td colspan="2">$3y\sin\frac{1}{2}(\alpha+\beta) + 2x\cos\frac{1}{2}(\alpha+\beta) = 6(\cos\alpha\cos\frac{1}{2}(\alpha+\beta) + \sin\alpha\sin\frac{1}{2}(\alpha+\beta))$
or
$3y\sin\frac{1}{2}(\alpha+\beta) + 2x\cos\frac{1}{2}(\alpha+\beta) = 6(\cos\beta\cos\frac{1}{2}(\alpha+\beta) + \sin\beta\sin\frac{1}{2}(\alpha+\beta))$</td><td></td></tr>
<tr><td></td><td colspan="2">$\frac{x}{3}\cos\frac{1}{2}(\alpha+\beta) + \frac{y}{2}\sin\frac{1}{2}(\alpha+\beta) = \cos\frac{1}{2}(\alpha-\beta)$ **ag**</td><td>A1cso</td></tr>
<tr><td></td><td colspan="2">This is cso – there must no errors in applying the factor formulae and sufficient working must be shown to justify the printed answer but allow
$\cos\beta\cos\frac{1}{2}(\alpha+\beta) + \sin\beta\sin\frac{1}{2}(\alpha+\beta) = \cos\frac{\alpha-\beta}{2}$</td><td></td></tr>
<tr><td></td><td colspan="2"></td><td>(4)</td></tr>
<tr><td>(b)</td><td colspan="2">$\left(\frac{3\cos\alpha + 3\cos\beta}{2}, \frac{2\sin\alpha + 2\sin\beta}{2}\right)$
or
$(3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha), 2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha))$
or
$(3\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta), 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta))$</td><td>B1</td></tr>
<tr><td></td><td colspan="2">Correct coordinates of mid-point in any form
Coordinates must be in this order but condone outer brackets missing</td><td></td></tr>
<tr><td></td><td colspan="2"></td><td>(1)</td></tr>
</table>

Question	Scheme		Marks
(c)	Centre of chord is $(3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha), 2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha))$ Attempt factor formulae on both coordinates of mid-point at any stage in (c) May be implied by their $\pm\frac{y}{x}$ below		M1
	$\pm\frac{y}{x}=\pm\frac{2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}{3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}\left(=\pm\frac{2\sin\frac{1}{2}(\beta+\alpha)}{3\cos\frac{1}{2}(\beta+\alpha)}\right)$ Or $\pm\frac{x}{y}=\pm\frac{3\cos\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}{2\sin\frac{1}{2}(\beta+\alpha)\cos\frac{1}{2}(\beta-\alpha)}\left(=\pm\frac{3\cos\frac{1}{2}(\beta+\alpha)}{2\sin\frac{1}{2}(\beta+\alpha)}\right)$		**dM1**A1
	M1: Obtains an expression for k or $-k$ or $\frac{1}{k}$ or $-\frac{1}{k}$ **Dependent on the previous M1 (factor formulae must have been used)** A1: Correct expression in any form		
	$m=-\frac{2\cos\frac{1}{2}(\beta+\alpha)}{3\sin\frac{1}{2}(\beta+\alpha)}$	Must be seen or used in (c)	B1
	$\frac{\sin\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta+\alpha)}=-\frac{2}{3m}$ So $\frac{y}{x}=\frac{2}{3}\left(-\frac{2}{3m}\right)\Rightarrow k=\frac{4}{9m}$		A1**cso**
			(5)
			Total 10

Question 3

2. (a)	Let $f(x) = 3 + x\sin\left(\frac{x}{4}\right)$ then f (13)= 1.593 [and f(14) = -1.911 need not be seen in (a)]	
	f(13.5) = -0.122, so root in [13, 13.5]	M1 A1
	f(13.25) = 0.746 so root in [13.25, 13.5]	A1 (3)
(b)	$\frac{\alpha - 13}{14 - \alpha} = \frac{1.593}{1.911}$ or $\frac{\alpha - 13}{1} = \frac{1.593}{1.593 + 1.911}$	M1 A1
	So $\alpha(1.911 + 1.593) = 1.593 \times 14 + 13 \times 1.911$ and $\alpha = \frac{47.145}{3.504} = 13.455$	dM1 A1 (4) **(7 marks)**

Question 4

Question Number	Scheme		Marks	
7.	(a) $\tan\theta = \sqrt{3}$ *or* $\sin\theta = \frac{\sqrt{3}}{2}$		M1	
	$\theta = \frac{\pi}{3}$	awrt 1.05	A1	**(2)**
	(b) $\frac{dx}{d\theta} = \sec^2\theta$, $\frac{dy}{d\theta} = \cos\theta$			
	$\frac{dy}{dx} = \frac{\cos\theta}{\sec^2\theta}$ $\left(= \cos^3\theta\right)$		M1 A1	
	At P, $m = \cos^3\left(\frac{\pi}{3}\right) = \frac{1}{8}$	Can be implied	A1	
	Using $mm' = -1$, $m' = -8$		M1	
	For normal $y - \frac{1}{2}\sqrt{3} = -8\left(x - \sqrt{3}\right)$		M1	
	At Q, $y = 0$ $-\frac{1}{2}\sqrt{3} = -8\left(x - \sqrt{3}\right)$			
	leading to $x = \frac{17}{16}\sqrt{3}$ $\left(k = \frac{17}{16}\right)$	1.0625	A1	**(6)**
	(c) $\int y^2\,dx = \int y^2 \frac{dx}{d\theta}\,d\theta = \int \sin^2\theta \sec^2\theta\,d\theta$		M1 A1	
	$= \int \tan^2\theta\,d\theta$		A1	
	$= \int \left(\sec^2\theta - 1\right)d\theta$		M1	
	$= \tan\theta - \theta$ $(+C)$		A1	
	$V = \pi\int_0^{\frac{\pi}{3}} y^2\,dx = \left[\tan\theta - \theta\right]_0^{\frac{\pi}{3}} = \pi\left[\left(\sqrt{3} - \frac{\pi}{3}\right) - (0 - 0)\right]$		M1	
	$= \sqrt{3}\pi - \frac{1}{3}\pi^2$ $\left(p = 1, q = -\frac{1}{3}\right)$		A1	**(7)**
				[15]

Question 5

Question Number	Scheme		Marks
Q3	$\sin x\dfrac{dy}{dx} - y\cos x = \sin 2x\sin x$		
	$\dfrac{dy}{dx} - \dfrac{y\cos x}{\sin x} = \dfrac{\sin 2x\sin x}{\sin x}$	An attempt to divide every term in the differential equation by $\sin x$. Can be implied.	M1
	$\dfrac{dy}{dx} - \dfrac{y\cos x}{\sin x} = \sin 2x$		
	Integrating factor $= e^{\int -\frac{\cos x}{\sin x}dx} = e^{-\ln\sin x}$	$e^{\int \pm\frac{\cos x}{\sin x}(dx)}$ or $e^{\int \pm \text{their } P(x)\,(dx)}$	dM1
		$e^{-\ln\sin x}$ or $e^{\ln\operatorname{cosec} x}$	A1 aef
	$= \dfrac{1}{\sin x}$	$\dfrac{1}{\sin x}$ or $(\sin x)^{-1}$ or $\operatorname{cosec} x$	A1 aef
	$\left(\dfrac{1}{\sin x}\right)\dfrac{dy}{dx} - \dfrac{y\cos x}{\sin^2 x} = \dfrac{\sin 2x}{\sin x}$		
	$\dfrac{d}{dx}\left(\dfrac{y}{\sin x}\right) = \sin 2x \times \dfrac{1}{\sin x}$	$\dfrac{d}{dx}(y \times \text{their I.F.}) = \sin 2x \times \text{their I.F}$	M1
	$\dfrac{d}{dx}\left(\dfrac{y}{\sin x}\right) = 2\cos x$	$\dfrac{d}{dx}\left(\dfrac{y}{\sin x}\right) = 2\cos x$ or $\dfrac{y}{\sin x} = \int 2\cos x\,(dx)$	A1
	$\dfrac{y}{\sin x} = \int 2\cos x\,dx$		
	$\dfrac{y}{\sin x} = 2\sin x + K$	A credible attempt to integrate the RHS with/without $+K$	dddM1
	$y = 2\sin^2 x + K\sin x$	$y = 2\sin^2 x + K\sin x$	A1 cao
			[8]

Question 6

Question Number	Scheme		Marks
Q4	$A = \frac{1}{2}\int_0^{2\pi} (a + 3\cos\theta)^2 \, d\theta$	Applies $\frac{1}{2}\int_0^{2\pi} r^2 (d\theta)$ with correct limits. Ignore $d\theta$.	B1
	$(a + 3\cos\theta)^2 = a^2 + 6a\cos\theta + 9\cos^2\theta$		
	$= \underline{a^2 + 6a\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right)}$	$\cos^2\theta = \frac{\pm 1 \pm \cos 2\theta}{2}$ Correct underlined expression.	M1 A1
	$A = \frac{1}{2}\int_0^{2\pi} \left(a^2 + 6a\cos\theta + \frac{9}{2} + \frac{9}{2}\cos 2\theta\right) d\theta$		
	$= \left(\frac{1}{2}\right)\left[a^2\theta + 6a\sin\theta + \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\right]_0^{2\pi}$	Integrated expression with at least 3 out of 4 terms of the form $\pm A\theta \pm B\sin\theta \pm C\theta \pm D\sin 2\theta$. Ignore the $\frac{1}{2}$. Ignore limits. $a^2\theta + 6a\sin\theta +$ correct ft integration. Ignore the $\frac{1}{2}$. Ignore limits.	M1* A1 ft
	$= \frac{1}{2}\left[\left(2\pi a^2 + 0 + 9\pi + 0\right) - (0)\right]$		
	$= \pi a^2 + \frac{9\pi}{2}$	$\pi a^2 + \frac{9\pi}{2}$	A1
	Hence, $\pi a^2 + \frac{9\pi}{2} = \frac{107}{2}\pi$	Integrated expression equal to $\frac{107}{2}\pi$.	dM1*
	$a^2 + \frac{9}{2} = \frac{107}{2}$		
	$a^2 = 49$		
	As $a > 0$, $a = 7$	$a = 7$	A1 cso **[8]**
	Some candidates may achieve $a = 7$ from incorrect working. Such candidates will not get full marks		