A-level Further Mathematics Further Pure 1 and 2
Paper E

1a. Express the complex number $-2 + (2\sqrt{3})\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta), -\pi < \theta \le \pi$ **[3]**

b. Solve the equation,

$$z^4 = -2 + (2\sqrt{3})\text{i}$$

Giving the roots in the form $r(\cos\theta + \text{i}\sin\theta), -\pi < \theta \le \pi$ **[5]**

(Total 8 marks)

2. The position vectors of the points A, B and C relative to an origin O are $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, 7\mathbf{i} - 3\mathbf{k}$ and $4\mathbf{i} + 4\mathbf{j}$ respectively.

Find,

a. $\overrightarrow{AC} \times \overrightarrow{BC}$ **[4]**

b. The area of triangle *ABC* **[2]**

c. An equation of the plane *ABC* in the form **r.n** = *p* **[2]**

(Total 8 marks)

3. Given that $y = arsinh\,(\tanh x)$, show that,

$$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\sqrt{1 + \tanh^2 x}}$$

[5]

(Total 5 marks)

4.

$$\mathbf{M} = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}$$

Where p and q are constants.

Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix **M**

a. Find the eigenvalue corresponding to this eigenvector, **[3]**

b. Find the value of p and the value of q. **[3]**

Given that 6 is another eigenvalue of **M,**

c. Find a corresponding eigenvector **[2]**

Given that $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is a third eigenvector of **M** with eigenvalue 3.

d. Find a matrix **P** and a diagonal matrix **D** such that,

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$ **[3]**

(Total 11 marks)

5. The curve C has polar equation

$$r = 1 + 2\cos\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point P on C, the tangent to C is parallel to the initial line.

Given that O is the pole, find the exact length of the line OP. **[7]**

(Total 7 marks)

6. Prove by induction that, for $n \in \mathbb{Z}^+$

a. $f(n) = 5^n + 8n + 3$ is divisible by 4. **[7]**

b. $\begin{pmatrix}3 & -2\\2 & -1\end{pmatrix}^n = \begin{pmatrix}2n+1 & -2n\\2n & 1-2n\end{pmatrix}$ **[7]**

(Total 14 marks)

7. $f(x) = (x^2 + 4)(x^2 + 8x + 25)$

a. Find the four roots of $f(x) = 0$ **[5]**

b. Find the sum of these four roots. **[2]**

(Total 7 marks)

End of Paper

Total Marks: 60

Mark Scheme

Question 1

3.			
(a)	$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$	B1	
	$\tan\theta = -\sqrt{3}$ (Also allow M mark for $\tan\theta = \sqrt{3}$)	M1	
	M mark can be implied by $\theta = \pm\frac{2\pi}{3}$ or $\theta = \pm\frac{\pi}{3}$		
	$\theta = \frac{2\pi}{3}$	A1	
			(3)
(b)	Finding the 4th root of their r: $r = 4^{1/4}$ $(= \sqrt{2})$	M1	
	For one root, dividing their θ by 4: $\theta = \frac{2\pi}{3} \div 4 = \frac{\pi}{6}$	M1	
	For another root, add or subtract a multiple of 2π to their θ **and** divide by 4 in correct order.	M1	
	$\sqrt{2}(\cos\theta + \mathrm{i}\sin\theta)$, where $\theta = -\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$	A1 A1	
			(5)
			8

Question 2

Question Number	Scheme	Marks
3. (a)	$\overrightarrow{AC} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{BC} = -3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ $\overrightarrow{AC} \times \overrightarrow{BC} = 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}$	B1, B1 M1 A1 (4)
(b)	Area of triangle $ABC = \frac{1}{2}\lvert 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}\rvert = \frac{1}{2}\sqrt{1225} = 17.5$	M1 A1 (2)
(c)	Equation of plane is $10x - 15y + 30z = -20$ or $2x - 3y + 6z = -4$ So $\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = -4$ or correct multiple	M1 A1 (2) (8 marks)

Question 3

Question Number	Scheme	Notes	Marks
1	$y = \text{arsinh}(\tanh x)$		
Way 1	$\sinh y = \tanh x$		B1
	$\cosh y \frac{dy}{dx} = \text{sech}^2 x$ or $\cosh y = \text{sech}^2 x \frac{dx}{dy}$	M1: $\pm\cosh y$ or $\pm\text{sech}^2 x$ A1: All correct	M1A1
	$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\cosh y}$		
	$\frac{dy}{dx} = \frac{\text{sech}^2 x}{\sqrt{1+\sinh^2 y}} = f(x)$	Uses a **correct identity** to express $\frac{dy}{dx}$ in terms of x only	M1
	$= \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}}$ *	cso. There must be no errors such as incorrect or missing or inconsistent variables and no missing h's.	A1*
			Total 5
Way 2	$t = \tanh x \Rightarrow y = \text{arsinh}t$	Replaces tanh$x$ by e.g. t	B1
	$\frac{dt}{dx} = \text{sech}^2 x, \frac{dy}{dt} = \frac{1}{\sqrt{1+t^2}}$	M1: $\frac{dt}{dx} = \pm\text{sech}^2 x, \frac{dy}{dt} = \pm\frac{1}{\sqrt{1+t^2}}$ A1: Correct $\frac{dt}{dx}$ and $\frac{dy}{dt}$ and correctly labelled	M1A1
	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\text{sech}^2 x}{\sqrt{1+t^2}} = f(x)$	Uses **correct form of the chain rule** for their variables to express $\frac{dy}{dx}$ in terms of x only	M1
	$= \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}}$ *	Cso. There must be no errors such as incorrect or missing or inconsistent variables and no missing h's.	A1*
			Total 5
Way 3	$u = \tanh x \Rightarrow \frac{du}{dx} = \text{sech}^2 x$	Correct derivative	B1
	$\int \frac{\text{sech}^2 x}{\sqrt{1+\tanh^2 x}} dx = \int \frac{\text{sech}^2 x}{\sqrt{1+u^2}} \frac{1}{\text{sech}^2 x} du$	M1: Complete substitution including the "dx" A1: Fully correct substitution	M1A1
	$= \int \frac{1}{\sqrt{1+u^2}} du = \text{arsinh}u (+c)$	Reaches arsinh$u$	M1
	$y = \text{arsinh}(\tanh x)(+c)$	Reaches $y = \text{arsinh}(\tanh x)$ with or without + c and no errors such as incorrect or missing or inconsistent variables or missing h's.	A1*
			Total 5

Question 4

Question Number	Scheme	Notes	Marks
6(a)	$\begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} p-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & q-\lambda \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	This statement is sufficient for this mark. May be implied by one correct equation e.g. $2p+4=2\lambda$, $-4-12-2=-2\lambda$, $4+q=\lambda$	M1
	$-4-12-2=-2\lambda \Rightarrow \lambda=9$	M1: Compares y-components to obtain a value for λ. Note that $-4-12-2=-2\lambda$ leading to a value for λ scores both method marks. If working is not clear, at least 2 terms of "$-4-12-2$" should be correct. A1: Correct eigenvalue	M1A1
			(3)
(b)	$\lambda=9 \Rightarrow 2p+4=18 \Rightarrow p=7$ $\lambda=9 \Rightarrow 4+q=9 \Rightarrow q=5$	M1: Uses their eigenvalue to form an equation in p or q A1: Either $p=7$ or $q=5$ A1: Both $p=7$ and $q=5$	M1A1A1
			(3)
(c)	$\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} 7x-2y=6x \\ -2x+6y-2z=6y \\ -2y+5z=6z \end{matrix}$		M1
	Uses the eigenvalue 6 and their value of p or q correctly to obtain at least 2 equations.		
	$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ or e.g. $\begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$	This vector or any multiple of this vector.	A1
	Note that an eigenvector can be found from the cross product of any 2 rows of **M** − 6**I** e.g. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}$		
			(2)

(d)	$\mathbf{P}=\begin{pmatrix}2 & "2" & 1\\ -2 & "1" & 2\\ 1 & "-2" & 2\end{pmatrix}$	Correct ft **P**. This should be a matrix of eigenvectors two of which are given in the question together with their eigenvector found from part (c). If an attempt is made to normalise the eigenvectors then allow the ft if slips are made when normalising.	B1ft
	$\mathbf{D}=\begin{pmatrix}"9" & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 3\end{pmatrix}$	Forms the matrix **D** by writing the eigenvalues 6, 3 and their λ on the leading diagonal and zeros elsewhere **or** attempts to calculate $\mathbf{P}^{\mathrm{T}}\mathbf{MP}$ to obtain a single 3 by 3 matrix. Consistency not needed for this mark.	M1
	$\left(\mathbf{P}=\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{pmatrix}, \mathbf{D}=\begin{pmatrix}9 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 3\end{pmatrix}\right)$ or $\left(\mathbf{P}=\begin{pmatrix}2 & -2 & 1\\ -2 & -1 & 2\\ 1 & 2 & 2\end{pmatrix}, \mathbf{D}=\begin{pmatrix}81 & 0 & 0\\ 0 & 54 & 0\\ 0 & 0 & 27\end{pmatrix}\right)$ Fully correct and consistent matrices		A1
	Note that the answers to part (d) may be implied e.g. $\mathbf{D}=\mathbf{P}^{\mathrm{T}}\mathbf{MP}=\begin{pmatrix}2 & -2 & 1\\ -2 & -1 & 2\\ 1 & 2 & 2\end{pmatrix}\begin{pmatrix}7 & -2 & 0\\ -2 & 6 & -2\\ 0 & -2 & 5\end{pmatrix}\begin{pmatrix}2 & -2 & 1\\ -2 & -1 & 2\\ 1 & 2 & 2\end{pmatrix}=\begin{pmatrix}81 & 0 & 0\\ 0 & 54 & 0\\ 0 & 0 & 27\end{pmatrix}$ Would score all 3 marks by implication.		
			(3)
			Total 11

Question 5

2.	$y = r\sin\theta = \sin\theta + 2\sin\theta\cos\theta$	B1
	$\frac{dy}{d\theta} = \cos\theta + 2\cos 2\theta$	M1
	$4\cos^2\theta + \cos\theta - 2 = 0$	A1oe
	$\cos\theta = \frac{-1 \pm \sqrt{1+32}}{8}$	M1 A1
	$OP = r = 1 + \frac{-1+\sqrt{1+32}}{4} = \frac{3+\sqrt{33}}{4}$	M1 A1
		(7)
		7

Question 6

Question Number	Scheme	Marks
Q8 (a)	$f(1) = 5 + 8 + 3 = 16$, (which is divisible by 4). ($\therefore$ True for $n = 1$).	B1
	Using the formula to write down $f(k+1)$, $f(k+1) = 5^{k+1} + 8(k+1) + 3$	M1 A1
	$f(k+1) - f(k) = 5^{k+1} + 8(k+1) + 3 - 5^k - 8k - 3$	M1
	$= 5(5^k) + 8k + 8 + 3 - 5^k - 8k - 3 = 4(5^k) + 8$	A1
	$f(k+1) = 4(5^k + 2) + f(k)$, **which is divisible by** 4	A1ft
	$\therefore$ True for $n = k + 1$ **if** true for $n = k$. True for $n = 1$, $\therefore$ true for all n.	A1cso (7)
(b)	For $n = 1$, $\begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1$ ($\therefore$ True for $n = 1$.)	B1
	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$	M1 A1 A1
	$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$	M1 A1
	$\therefore$ True for $n = k + 1$ **if** true for $n = k$. **True for** $n = 1$, $\therefore$ **true for all** n	A1 cso (7) [14]

Question 7

Question Number	Scheme	Marks
Q3 (a)	$x^2+4=0 \quad \Rightarrow \quad x=ki, \quad x=\pm 2i$	M1, A1
	Solving 3-term quadratic	M1
	$x=\dfrac{-8\pm\sqrt{64-100}}{2} \quad =-4+3i$ and $-4-3i$	A1 A1ft
		(5)
(b)	$2i+(-2i)+(-4+3i)+(-4-3i)=-8$	M1 A1cso
		(2)
		[7]
	Alternative method : Expands f(x) as quartic and chooses $\pm$ coefficient of x^3	M1
	-8	A1 cso