A-level Further Mathematics Further Pure 1 and 2 Paper A

1.

$$z = -2 + (2\sqrt{3})i$$

a. Find the modulus and the argument of z **[3]**

Using de Moivre's theorem,

b. Find z^6, simplifying your answer. **[2]**

c. Find the values of w such that $w^4 = z^3$, giving your answers in the form $a + ib$ where $a, b \in R$ **[4]**

(9 marks)

2.

(i) A group *G* contains distinct elements *a*, *b* and *e* where *e* is the identity element, and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$ **[4]**

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15.

(*a*) Find the order of each element of *H*. **[3]**

(*b*) Find three subgroups of *H* each of order 4, and describe each of these subgroups. **[4]**

The elements of another group *J* are the matrices $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$

where $k = 1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(*c*) Determine whether *H* and *J* are isomorphic, giving a reason for your answer. **[2]**

(13 marks)

3.

$$z = 2 - i\sqrt{3}$$

a. Calculate arg z, giving your answer in radians to 2 decimal places. **[2]**

Use algebra to express,

b. $z + z^2$ in the form $a + bi\sqrt{3}$, where a and b are integers. **[3]**

c. $\frac{z+7}{z-1}$ in the form $c + di\sqrt{3}$, where c and d are integers. **[4]**

Given that,

$$w = \gamma - 3\mathrm{i}$$

Where γ is a real constant and $\arg(4 - 5\mathrm{i} + 3w) = -\frac{\pi}{2}$.

d. Find the value of γ **[2]**

(11 marks)

4a. Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that,

$$\sum_{r=1}^{n}(r^3 + 6r - 3) = \frac{1}{4}n^2(n^2 + 2n + 13)$$

For all positive integers n **[5]**

b. Hence find the exact value of,

$$\sum_{r=16}^{30}(r^3 + 6r - 3)$$

[2]

(7 marks)

5. Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$$

a. Find the characteristic equation of the matrix **A**.

[2]

b. Hence show that $\mathbf{A}^3 = 43\mathbf{A} - 42\mathbf{I}$

[3]

(5 marks)

6. A sequence of numbers is defined by

$$u_1 = 8u_{n+1} = 4u_n - 9n, \quad n \geq 1$$

Prove by induction that, for $n \in \mathrm{R}$, $u_n = 4^n + 3n + 1$ **[3]**

(3 marks)

7. The line l_1 has vector equation,

$$r = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and the line l_2 has vector equation,

$$r = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

Where, λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ.

a. Write down the coordinates of A. **[1]**

b. Find the value of $\cos\theta$. **[3]**

The point X lies on l_1 where $\lambda = 4$.

c. Find the coordinates of X **[1]**

d. Find the vector $\overrightarrow{AX}$ **[2]**

e. Hence, or otherwise, show that $|\overrightarrow{AX}| = 4\sqrt{26}$ **[2]**

The point Y lies on l_2. Given that the vector $\overrightarrow{YX}$ is perpendicular to l_1,

f. Find the length of AY, giving your answer to 3 significant figures. **[3]**

(12 marks)

End of Paper

Total Marks: 60

Mark Scheme

Question 1

Question Number	Scheme	Marks
2 (a)	$\lvert z\rvert = 4$	B1
	$\arg z = \arctan\left(\frac{-2\sqrt{3}}{2}\right) = \arctan\left(-\sqrt{3}\right) = \frac{2\pi}{3}$ or 120°	M1A1 (3)
(b)	$z^6 = \left(4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)\right)^6 = 4^6\left(\cos 4\pi + \mathrm{i}\sin 4\pi\right)$ or $z^6 = \left(4\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^6$	M1
	$= 4096$ or 4^6 or 2^{12}	A1 cso (2)
	(a) and (b) can be marked together	
(c)	$z^{\frac{3}{4}} = 4^{\frac{3}{4}}\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)^{\frac{3}{4}} = 4^{\frac{3}{4}}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$	
	$w = \mathrm{i}2\sqrt{2}$ oe or any other correct root	B1
	$4^{\frac{3}{4}}\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + \mathrm{i}\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)^{\frac{3}{4}}$	M1
	$(n = 0$ see above$)$	
	$n = 1$ $w = 2\sqrt{2}$ oe	
	$n = 2$ $w = -\mathrm{i}2\sqrt{2}$ oe	
	$n = 3$ $w = -2\sqrt{2}$ oe	A1A1 (4)
		[9]

Question 2

Question	Scheme	Marks	AOs
4(i)	If we assume $ab = ba$; as $a^2b = ba$ then $ab = a^2b$	M1	2.1
	So $a^{-1}abb^{-1} = a^{-1}a^2bb^{-1}$	M1	2.1
	So $e = a$	A1	2.2a
	But this is a contradiction, as the elements e and a are distinct so $ab \neq ba$	A1	2.4
		(4)	
(ii)(a)	2 has order 4 and 4 has order 2	M1	1.1b
	7, 8 and 13 have order 4	A1	1.1b
	11 and 14 have order 2 and 1 has order 1	A1	1.1b
		(3)	
(ii)(b)	Finds the subgroup {1, 2, 4, 8} or the subgroup {1, 7, 4, 13}	M1	1.1b
	Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7	A1	2.4
	Finds {1, 4, 11, 14}	B1	2.2a
	States each element has order 2 or refers to it as Klein Group	B1	2.5
		(4)	
(ii)(c)	J has an element of order 8, (H does not) or J is a cyclic group (H is not) or other valid reason	M1	2.4
	They are not isomorphic	A1	2.2a
		(2)	
			(13 marks)

Question 3

Question Number	Scheme	Notes	Marks
7. (a)	$\arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$	$\tan^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$ or $\tan^{-1}\left(\pm\frac{2}{\sqrt{3}}\right)$ seen or evaluated	M1
	Awrt ±0.71 or awrt ±0.86 can be taken as evidence for the method mark. Or ±40.89 or ±49.10 if working in degrees		
	$= -0.7137243789.. = -0.71$ (2 dp)	awrt -0.71 or awrt 5.57	A1
	NB $\tan\left(\frac{\sqrt{3}}{2}\right) = 1.18$ and $\tan\left(\frac{2}{\sqrt{3}}\right) = 2.26$ **and both score M0**		
			[2]
(b)	$z^2 = \left(2 - i\sqrt{3}\right)\left(2 - i\sqrt{3}\right)$ $= 4 - 2i\sqrt{3} - 2i\sqrt{3} + 3i^2$	An attempt to multiply out the brackets to give four terms (or four terms implied).	M1
	$= 2 - i\sqrt{3} + \left(4 - 4i\sqrt{3} - 3\right)$ $= 2 - i\sqrt{3} + \left(1 - 4i\sqrt{3}\right)$ $= 3 - 5i\sqrt{3}$ (Note: $a = 3, b = -5$.)	M1: An understanding that $i^2 = -1$ and an attempt to add z and put in the form $a + bi\sqrt{3}$ A1: $3 - 5i\sqrt{3}$	M1A1
	$z + z^2 = 2 - i\sqrt{3} + \left(4 - 4i\sqrt{3} + 3\right) = 9 - 5i\sqrt{3}$ scores M1M0A0 (No evidence of $i^2 = -1$)		
			[3]
(c)	$\frac{z+7}{z-1} = \frac{2 - i\sqrt{3} + 7}{2 - i\sqrt{3} - 1}$	Substitutes $z = 2 - i\sqrt{3}$ into both numerator and denominator.	M1
	$= \frac{\left(9 - i\sqrt{3}\right)}{\left(1 - i\sqrt{3}\right)} \times \frac{\left(1 + i\sqrt{3}\right)}{\left(1 + i\sqrt{3}\right)}$	Simplifies $\frac{z+7}{z-1}$ **and** multiplies by $\frac{\textbf{their } \left(1 + i\sqrt{3}\right)}{\textbf{their } \left(1 + i\sqrt{3}\right)}$	dM1
	$= \frac{9 + 9i\sqrt{3} - i\sqrt{3} + 3}{1 + 3}$ $= \frac{12 + 8i\sqrt{3}}{4}$	Simplifies realising that a real number is needed in the denominator and applies $i^2 = -1$ in their numerator expression and denominator expression.	M1
	$= 3 + 2i\sqrt{3}$ (Note: $c = 3, d = 2$.)	$3 + 2i\sqrt{3}$	A1
			[4]
(d)	$w = \lambda - 3i$, and $\arg(4 - 5i + 3w) = -\frac{\pi}{2}$		
	$(4 - 5i + 3w = 4 + 3\lambda - 14i)$		
	So real part of $(4 - 5i + 3w) = 0$ or $4 + 3\lambda = 0$	States real part of $(4 - 5i + 3w) = 0$ or $4 + 3\lambda = 0$	M1
	So, $\lambda = -\frac{4}{3}$	$-\frac{4}{3}$	A1
			[2]
	Allow $\pm\left(\frac{14}{3\lambda + 4}\right) = \pm\infty \Rightarrow 3\lambda + 4 = 0$ M1 $\Rightarrow \lambda = -\frac{4}{3}$ A1		
			11 marks

Question 4

Question Number	Scheme	Notes	Marks
4. (a)	$\sum_{r=1}^{n}(r^3 + 6r - 3)$		
	$= \underline{\frac{1}{4}n^2(n+1)^2 + 6.\frac{1}{2}n(n+1)} - 3n$	M1; An attempt to use at least one of the standard formulae correctly in summing at least 2 terms of $r^3 + 6r - 3$ A1: Correct underlined expression. B1: $-3 \to -3n$	M1A1B1
	$= \frac{1}{4}n^2(n+1)^2 + 3n^2 + 3n - 3n$		
	If any marks have been lost, no further marks are available in part (a)		
	$= \frac{1}{4}n^2(n+1)^2 + 3n^2$ $= \frac{1}{4}n^2\left((n+1)^2 + 12\right)$	Cancels out the $3n$ and attempts to factorise out at least $\frac{1}{4}n$.	dM1
	$= \frac{1}{4}n^2\left(n^2 + 2n + 13\right)$ **(AG)**	Correct answer **with no errors seen.**	A1 *
	Provided the first 3 marks are scored, allow the next two marks for correctly showing the algebraic equivalence. E.g. showing that both $\frac{1}{4}n^2(n+1)^2 + 6.\frac{1}{2}n(n+1) - 3n$ and $\frac{1}{4}n^2\left(n^2 + 2n + 13\right) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{13}{4}n^2$		
	There are no marks for proof by induction but apply the scheme if necessary.		
			[5]
(b)	$S_n = \sum_{r=16}^{30}(r^3 + 6r - 3) = S_{30} - S_{15}$		
	$= \frac{1}{4}(30)^2\left(30^2 + 2(30) + 13\right) - \frac{1}{4}(15)^2\left(15^2 + 2(15) + 13\right)$	Use of $S_{30} - S_{15}$ or $S_{30} - S_{16}$	M1
	NB They must be using $S_n = \frac{1}{4}n^2\left(n^2 + 2n + 13\right)$ **not** $S_n = n^3 + 6n - 3$		
	$= 218925 - 15075$		
	$= 203850$	203850	A1 **cao**
	NB $S_{30} - S_{16} = 218925 - 19264 = 199661$ (Scores M1 A0)		
			[2]
			7 marks

Question 5

Question	Scheme	Marks	AOs
6(a)	Consider $\det\begin{pmatrix}3-\lambda & 1\\ 6 & 4-\lambda\end{pmatrix}=(3-\lambda)(4-\lambda)-6$	M1	1.1b
	So $\lambda^2-7\lambda+6=0$ is characteristic equation	A1	1.1b
		(2)	
	So $\mathbf{A}^2=7\mathbf{A}-6\mathbf{I}$	B1ft	1.1b
(b)	Multiplies both sides of their equation by **A** so $\mathbf{A}^3=7\mathbf{A}^2-6\mathbf{A}$	M1	3.1a
	Uses $\mathbf{A}^3=7(7\mathbf{A}-6\mathbf{I})-6\mathbf{A}$ So $\mathbf{A}^3=43\mathbf{A}-42\mathbf{I}$ *	A1*cso	1.1b
		(3)	
			(5 marks)

Notes:

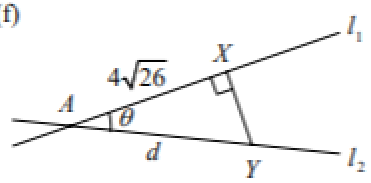
(a)
M1: Complete method to find characteristic equation
A1: Obtains a correct three term quadratic equation – may use variable other than λ

(b)
B1ft: Uses Cayley Hamilton Theorem to produce equation replacing λ with **A** and constant term with constant multiple of identity matrix, **I**
M1: Multiplies equation by **A**
A1*: Replaces $\mathbf{A}^2$ by linear expression in **A** and achieves printed answer with no errors

Question 6

$u_1 = 8$ $n = 1$ Assume true for n = k so that $u_k = 4^k + 3k + 1$	**B1**
$u_{k+1} = 4\left(4^k + 3k + 1\right) - 9k$ $= 4^{k+1} + 12k + 4 - 9k$ $= 4^{k+1} + 3k + 4$ $= 4^{k+1} + 3(k+1) + 1$	**A1**
If true for $n = k$, then true for $n = k + 1$ and as true for $n = 1$, true for all n	**A1**

Question 7

Question Number	Scheme	Marks
Q4	(a) A: $(-6, 4, -1)$ — Accept vector forms	B1 (1)
	(b) $\begin{pmatrix}4\\-1\\3\end{pmatrix}.\begin{pmatrix}3\\-4\\1\end{pmatrix} = 12+4+3 = \sqrt{4^2+(-1)^2+3^2}\sqrt{3^2+(-4)^2+1^2}\cos\theta$	M1 A1
	$\cos\theta = \frac{19}{26}$ — awrt 0.73	A1 (3)
	(c) X: $(10, 0, 11)$ — Accept vector forms	B1 (1)
	(d) $\overrightarrow{AX} = \begin{pmatrix}10\\0\\11\end{pmatrix} - \begin{pmatrix}-6\\4\\-1\end{pmatrix}$ — Either order	M1
	$= \begin{pmatrix}16\\-4\\12\end{pmatrix}$ — cao	A1 (2)
	(e) $\lvert\overrightarrow{AX}\rvert = \sqrt{16^2+(-4)^2+12^2}$	M1
	$= \sqrt{416} = \sqrt{16\times26} = 4\sqrt{26}$ ✱ — Do not penalise if consistent incorrect signs in (d)	A1 (2)
	(f)  Use of correct right angled triangle	M1
	$\frac{\lvert\overrightarrow{AX}\rvert}{d} = \cos\theta$	M1
	$d = \frac{4\sqrt{26}}{\frac{19}{26}} \approx 27.9$ — awrt 27.9	A1 (3)
		[12]