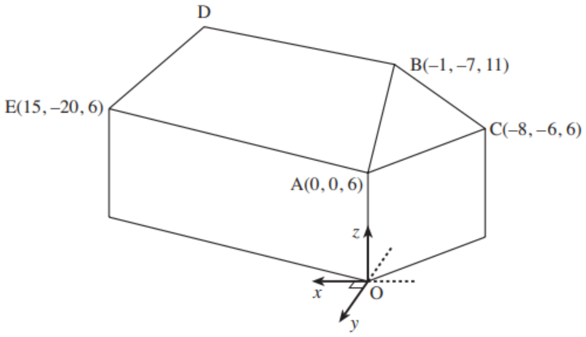
Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Equation of a Line/Plane in Three Dimensions (1) Chapter Reference: Core Pure 1, Chapter 9	10 minutes

1. Fig. 1 illustrates a house. All units are in metres. The coordinates of A, B, C and E are as shown. BD is horizontal and parallel to AE.



a. Find the length AE. **(2)**

b. Find a vector equation of the line BD. Given that the length of BD is 15 metres, find the coordinates of D. **(4)**

c. Verify that the equation of the plane ABC is $-3x + 4y + 5z = 30$. Give a vector normal to this plane. **(4)**

Solutions

1a.

$AE = \sqrt{(15^2 + 20^2 + 0^2)} = 25$	**M1A1**

1b.

$\overrightarrow{AE} = \begin{pmatrix} 15 \\ -20 \\ 0 \end{pmatrix} = 5\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ Any correct form	**M1**
Equation of BD is $\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 15 \\ -20 \\ 0 \end{pmatrix}$	**A1**
$BD = 15 \Rightarrow \lambda = 3$ $\lambda = 3$ or $3/5$ as appropriate	**M1**
$\Rightarrow$ D is $(8, -19, 11)$ cao	**A1**

1c.

One verification	**M1**
At A: $-3 \times 0 + 4 \times 0 + 5 \times 6 = 30$ At B: $-3 \times (-1) + 4 \times (-7) + 5 \times 11 = 30$ At C: $-3 \times (-8) + 4 \times (-6) + 5 \times 6 = 30$ (OR B1 Normal, M1 scalar product with 1 vector in the plane, A1two correct, A1 verification with a point	**A2,1,0**
Normal is $\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$ OR M1 vector form of equation of plane eg r=0i+0j+6k+μ(i+7j-5k)+ν(8i+6j+0k) M1 elimination of both parameters A1 equation of plane B1 Normal *)	**B1**

Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Scalar Product (2)

Chapter Reference: Core Pure 1, Chapter 9

10 minutes

1. Relative to a fixed origin O, the point A has position vector $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, the point B has position vector $5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, and the point C has position vector $7\boldsymbol{i} - \boldsymbol{j}$. Find the cosine of angle ABC. **(4)**

2. A triangle ABC has vertices $A(-2, 4, 1)$, $B(2, 3, 4)$ and $C(4, 8, 3)$. By calculating a suitable scalar product, show that angle ABC is a right angle. Hence calculate the area of the triangle. **(6)**

Solutions

1.

$\overrightarrow{AB} = (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, $\overrightarrow{CB} = (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, (or $\overrightarrow{BA}$, $\overrightarrow{BC}$, or $\overrightarrow{AB}$, $\overrightarrow{BC}$ stated in above form or column vector form.	**M1A1**
$\cos A\hat{B}C = \frac{CB \bullet AB}{\lvert CB \rvert \lvert AB \rvert} = -\frac{4}{9}$ or (-0.444)	**M1A1**

2.

$\overline{BA} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$, $\overline{BC} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ soi, condone wrong sense	**B1**
$\overline{BA}.\overline{BC} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = (-4)\times 2 + 1 \times 5 + (-3)\times(-1)$ scalar product	**M1**
$= -8 + 5 + 3 = 0$ $\Rightarrow$ angle ABC = 90° ($= 0$)	**A1**
Area of triangle = ½ × BA × BC area of triangle formula oe	**M1**
$= \frac{1}{2} \times \sqrt{(-4)^2 + 1^2 + 3^2} \times \sqrt{2^2 + 5^2 + -1)^2}$ $= ½ \times \sqrt{26} \times \sqrt{30}$ length formula	**M1**
$= 13.96$ sq units accept 14.0 and $\sqrt{195}$	**A1**

Further Maths A-Level Starter Activity

Topic: Calculating Angles Between Lines and Planes (3) Chapter Reference: Core Pure 1, Chapter 9	**10 minutes**

1. a. Find a vector equation of the line *l* joining the points $(0, 1, 3)$ and $(-2, 2, 5)$. **(2)**

   b. Find the point of intersection of the line *l* with the plane $x + 3y + 2z = 4$. **(3)**

   c. Find the acute angle between the line *l* and the normal to the plane. **(3)**

Solutions

1a.

$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \ldots$ need **r** (or another letter) = or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for first B1	**B1**
$\ldots + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ **NB answer is not unique** eg $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ Accept **i**/**j**/**k** form and condone row vectors.	**B1**

1b.

$x + 3y + 2z = 4$ $\Rightarrow \; -2\lambda + 3(1+\lambda) + 2(3+2\lambda) = 4$ substituting their line in plane equation (condone a slip if intention clear)	**M1**
$\Rightarrow \; 5\lambda = -5, \; \lambda = -1$ www cao **NB λ is not unique** as depends on choice of line in (i)	**A1**
so point of intersection is (2, 0, 1) www cao	**A1**

1c.

Angle between $-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is θ where …	**M1**
$\cos\theta = \dfrac{-2\times 1 + 1\times 3 + 2\times 2}{\sqrt{9}\sqrt{14}} = \dfrac{5}{3\sqrt{14}}$ correct formula (including cosine) , with substitution, for these vectors	**M1**
$\Rightarrow \; \theta = 63.5°$ www cao (63.5 in degrees (or better) or 1.109 radians or better)	**A1**

Further Maths A-Level Starter Activity

Topic: Points of Intersection (4)

Chapter Reference: Core Pure 1, Chapter 9

10 minutes

1. Determine whether the lines $\frac{x-1}{1}=\frac{y+2}{-1}=\frac{z+4}{2}$ and $\frac{x+3}{2}=\frac{y-1}{3}=\frac{z-5}{4}$ intersect or are skew. **(5)**

2. A line l has equation $\boldsymbol{r}=3\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}+t(\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k})$ and a plane Π has equation $8x-7y+10z=7$. Determine whether l lies in Π, is parallel to Π without intersecting it, or intercepts Π at one point. **(5)**

Solutions

1. Either

For correct parametric form for either line	**B1**
For 3 equations using 2 different parameters	**M1**
$\left.\begin{array}{l} 1+s=-3+2t \\ -2-s=1+3t \\ -4+2s=5+4t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} s-2t=-4 \\ s+3t=-3 \\ 2s-4t=9 \end{array}\right.$	**A1**
For attempting to solve to show (in)consistency	**M1**
$\Rightarrow$ contradiction, so skew For correct conclusion	**A1**

Or

line segment between l_1 and l_2 = $\pm[4,-3,-9]$ For correct vector	**B1**
$\mathbf{n}=[1,-1,2]\times[2,3,4]=(\pm)[-2,0,1]$ For finding vector product of direction vectors	**M1**
distance $=\dfrac{\lvert[4,-3,-9].[-2,0,1]\rvert}{\left(\sqrt{2^2+0^2+1^2}\right)}=\dfrac{17}{(\sqrt{5})}$	**A1**
For using numerator of distance formula	**M1**
$\neq 0$, so skew For correct scalar product and correct conclusion	**A1**

2. Either

$(\mathbf{r}=)[3+t,1+4t,-2+2t]$ For parametric form of l seen or implied	**M1**
$8(3+t)-7(1+4t)+10(-2+2t)=7$ For substituting into plane equation	**M1A1**
$\Rightarrow (0t)+(-3)=7 \Rightarrow$ contradiction For obtaining a contradiction	**A1**
l is parallel to Π, no intersection For conclusion from correct working	**B1**

Or

$[1,4,2].[8,-7,10]=0$ For finding scalar product of direction vectors	**M1**
$\Rightarrow l$ is parallel to Π For correct conclusion	**A1**
$(3,1,-2)$ into Π For substituting point into plane equation	**M1**
$\Rightarrow 24-7-20\neq 7$ For obtaining a contradiction	**A1**
l is parallel to Π, no intersection For conclusion from correct working	**B1**

Or

Solve $\dfrac{x-3}{1}=\dfrac{y-1}{4}=\dfrac{z+2}{2}$ and $8x-7y+10z=7$ For eliminating one variable	**M1A1**
For eliminating another variable	**M1**
eg $4z+4=4z+8$ For obtaining a contradiction	**A1**
l is parallel to Π, no intersection For conclusion from correct working	**B1**

Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Finding Perpendiculars Chapter Reference: Core Pure 2, Chapter 9	**10 minutes**

1. Two skew lines have equations $\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3}$ and $\frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}$.

   a. Find the direction of the common perpendicular to the lines. **(2)**

   b. Find the shortest distance between the lines. **(4)**

2. The plane Π_1 has vector equation $\boldsymbol{r}.(3\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}) = 5$. Find the perpendicular distance from the point $(6, 2, 12)$ to the plane Π_1. **(3)**

Jethwa Maths

Solutions

1. a

$\mathbf{n} = [2, 1, 3] \times [3, 1, 5]$ For using direction vectors and attempt to find vector product	**M1**
$= [2, -1, -1]$ For correct direction (allow multiples)	**A1**

b

$d = \dfrac{\lvert[5, 2, 1] . [2, -1, -1]\rvert}{\sqrt{6}}$ For $(\mathbf{AB} =) [5, 2, 1]$ or any vector joining lines	**B1**
For attempt at evaluating $\mathbf{AB} . \mathbf{n}$	**M1**
For $\lvert\mathbf{n}\rvert$ in denominator	**M1**
$= \dfrac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6} = 2.8577$ For correct distance	**A1**

2. Either

$(\mathbf{6i} + \mathbf{2j} + \mathbf{12k}).(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 34$ Attempt scalar product	**M1**
$\left\lvert \dfrac{(\mathbf{6i} + \mathbf{2j} + \mathbf{12k}).(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) - 5}{\sqrt{3^2 + 4^2 + 2^2}} \right\rvert$ Use of correct formula	**M1**
$\sqrt{29}$ (not $-\sqrt{29}$) Correct distance (Allow $29/\sqrt{29}$)	**A1**

Or

$\mathbf{r} = (\mathbf{6i} + \mathbf{2j} + \mathbf{12k}) + \lambda(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ $\therefore\ 6+3\lambda\ \ 3+\ \ 2-4\lambda\ \ -4\ +\ 12+2\lambda\ \ 2 = 5$ Substitutes the parametric coordinates of the line through (6, 2, 12) perpendicular to the plane into the cartesian equation.	**M1**
$\lambda = -1 \Rightarrow\ 3, 6, 10$ or $-3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ Solves for λ to obtain the required point or vector.	**M1**
$\sqrt{29}$ Correct distance	**A1**

Or

Parallel plane containing (6, 2, 12) is $\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 34$ $\Rightarrow \dfrac{\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})}{\sqrt{29}} = \dfrac{34}{\sqrt{29}}$ Origin to this plane is $\dfrac{34}{\sqrt{29}}$	**M1**
$\Rightarrow \dfrac{\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})}{\sqrt{29}} = \dfrac{5}{\sqrt{29}}$ Origin to plane is $\dfrac{5}{\sqrt{29}}$	**M1**
$\dfrac{34}{\sqrt{29}} - \dfrac{5}{\sqrt{29}} = \sqrt{29}$ Correct distance	**A1**

Further Maths A-Level Starter Activity

Topic: Finding Perpendiculars (5)

Chapter Reference: Core Pure 1, Chapter 9

10 minutes

1. Two skew lines have equations $\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3}$ and $\frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}$.
   a. Find the direction of the common perpendicular to the lines. **(2)**
   b. Find the shortest distance between the lines. **(4)**

2. The plane Π_1 has vector equation $\boldsymbol{r}.(3\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}) = 5$. Find the perpendicular distance from the point $(6, 2, 12)$ to the plane Π_1. **(3)**

Solutions

1a.

$\mathbf{n} = [2, 1, 3] \times [3, 1, 5]$ For using direction vectors and attempt to find vector product	**M1**
$= [2, -1, -1]$ For correct direction (allow multiples)	**A1**

1b.

$d = \dfrac{\lvert [5, 2, 1] . [2, -1, -1] \rvert}{\sqrt{6}}$ For $(\mathbf{AB} =) [5, 2, 1]$ or any vector joining lines	**B1**
For attempt at evaluating $\mathbf{AB} . \mathbf{n}$	**M1**
For $\lvert \mathbf{n} \rvert$ in denominator	**M1**
$= \dfrac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6} = 2.8577$ For correct distance	**A1**

2. Either

$(6\mathbf{i}+2\mathbf{j}+12\mathbf{k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = 34$ Attempt scalar product	**M1**
$\left\lvert \dfrac{(6\mathbf{i}+2\mathbf{j}+12\mathbf{k}).(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})-5}{\sqrt{3^2+4^2+2^2}} \right\rvert$ Use of correct formula	**M1**
$\sqrt{29}$ (not $-\sqrt{29}$) Correct distance (Allow $29/\sqrt{29}$)	**A1**

Or

$\mathbf{r} = (6\mathbf{i}+2\mathbf{j}+12\mathbf{k}) + \lambda(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})$ $\therefore\ 6+3\lambda\ 3+\ 2-4\lambda\ \ -4\ +\ 12+2\lambda\ 2 = 5$ Substitutes the parametric coordinates of the line through (6, 2, 12) perpendicular to the plane into the cartesian equation.	**M1**
$\lambda = -1 \Rightarrow\ 3, 6, 10$ or $-3\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ Solves for λ to obtain the required point or vector.	**M1**
$\sqrt{29}$ Correct distance	**A1**

Or

Parallel plane containing (6, 2, 12) is $\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = 34$ $\Rightarrow \dfrac{\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})}{\sqrt{29}} = \dfrac{34}{\sqrt{29}}$ Origin to this plane is $\dfrac{34}{\sqrt{29}}$	**M1**
$\Rightarrow \dfrac{\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})}{\sqrt{29}} = \dfrac{5}{\sqrt{29}}$ Origin to plane is $\dfrac{5}{\sqrt{29}}$	**M1**
$\dfrac{34}{\sqrt{29}} - \dfrac{5}{\sqrt{29}} = \sqrt{29}$ Correct distance	**A1**