Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Proof by Induction (1)

Chapter Reference: Core Pure 1, Chapter 8

10 minutes

1. Prove by induction that, for $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

2. Prove by induction that, for $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

(5)

Solutions

1.

When $n = 1$, LHS $= \frac{1}{1\times 2} = \frac{1}{2}$, RHS $= \frac{1}{1+1} = \frac{1}{2}$. So LHS = RHS and result true for $n = 1$	**B1** **M1**
Assume true for $n = k$; $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$ and so $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$	**M1**
$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$	**M1** **A1**
and so result is true for $n = k + 1$ (and by induction true for $n \in \mathbf{Z}^+$)	**B1**

2.

If $n = 1$, $\sum_{r=1}^{n} r(r+3) = 1\times 4 = 4$ and $\frac{1}{3}n(n+1)(n+5) = \frac{1}{3}\times 1\times 2\times 6 = 4$,	**B1**
(so true for $n = 1$. Assume true for $n = k$) So $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$ $= \frac{1}{3}(k+1)[k(k+5)+3(k+4)] = \frac{1}{3}(k+1)[k^2+8k+12]$	**M1**
$= \frac{1}{3}(k+1)(k+2)(k+6)$ which implies is true for $n = k+1$	**A1**
As result is true for $n = 1$ this implies true for all positive integers and so result is true by induction	**M1** **A1**

Further Maths A-Level Starter Activity

Topic: Proof by Induction (2)

Chapter Reference: Core Pure 1, Chapter 8

10 minutes

1. Prove by induction, that for $n \in \mathbb{Z}^+$,

   $f(n) = 5^n + 8n + 3$ is divisible by 4. **(7)**

2. $f(n) = 2^n + 6^n$,

   Show that $f(k+1) = 6f(k) - 4(2^k)$ **(3)**

Solutions

1.

$f(1) = 5 + 8 + 3 = 16$, (which is divisible by 4). ($\therefore$ True for $n = 1$).	**B1**
Using the formula to write down $f(k + 1)$, $f(k+1) = 5^{k+1} + 8(k+1) + 3$	**M1** **A1**
$f(k+1) - f(k) = 5^{k+1} + 8(k+1) + 3 - 5^k - 8k - 3$ $= 5(5^k) + 8k + 8 + 3 - 5^k - 8k - 3 = 4(5^k) + 8$	**M1** **A1**
$f(k+1) = 4(5^k + 2) + f(k)$, **which is divisible by** 4	**A1**
$\therefore$ True for $n = k + 1$ **if** true for $n = k$. True for $n = 1$, $\therefore$ true for all n.	**A1**

2. Either LHS:

LHS = $f(k+1) = 2^{k+1} + 6^{k+1}$	**M1**
$= 2(2^k) + 6(6^k)$	**A1**
$= 6(2^k + 6^k) - 4(2^k) = 6f(k) - 4(2^k)$	**A1**

Either RHS

$= 6f(k) - 4(2^k) = 6(2^k + 6^k) - 4(2^k)$	**M1**
$= 2(2^k) + 6(6^k)$	**A1**
$= 2^{k+1} + 6^{k+1} = f(k+1)$	**A1**

or,

$f(k+1) - 6f(k) = 2^{k+1} + 6^{k+1} - 6(2^k + 6^k)$	**M1**
$= (2-6)(2^k) = -4.2^k$	**A1**
and so $f(k+1) = 6f(k) - 4(2^k)$	**A1**

Further Maths A-Level Starter Activity

Topic: Proof by Induction (3)

Chapter Reference: Core Pure 1, Chapter 8

7 minutes

1. Prove by induction, that for $n \in \mathbb{Z}^+$:

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$$

(7)

Solutions

1.

For $n = 1$, $\begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1$ ($\therefore$ True for $n = 1$.)	**B1**
$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$	**M1** **A1** **A1**
$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$	**M1** **A1**
$\therefore$ True for $n = k + 1$ **if** true for $n = k$. **True for n = 1, $\therefore$ true for all n**	**A1**

Further Maths A-Level Starter Activity

Topic: Proof by Induction (4)

Chapter Reference: Core Pure 1, Chapter 8

10 minutes

1. A series of positive integers $u_1, u_2, u_3, \ldots$ is defined by

   $u_1 = 6$ and $u_{n+1} = 6u_n - 5$ for $n \geq 1$

   Prove by induction that $u_n = 5 \times 6^{n-1} + 1$ for $n \geq 1$ **(5)**

2. A sequence of positive integers is defined by

   $u_1 = 1$ and $u_{n+1} = u_n + n(3n + 1), \quad n \in \mathbb{Z}^+$

   Prove by induction that $u_n = n^2(n - 1) + 1, \quad n \in \mathbb{Z}^+$ **(5)**

Solutions

1.

At $n=1$, $u_n = 5\times 6^0 + 1 = 6$ and so result true for $n = 1$	**B1**
Assume true for $n = k$; $u_k = 5\times 6^{k-1} + 1$,and so $u_{k+1} = 6(5\times 6^{k-1}+1) - 5$	**M1** **A1**
$\therefore u_{k+1} = 5\times 6^k + 6 - 5 \quad \therefore u_{k+1} = 5\times 6^k + 1$	**A1**
and so result is true for $n = k + 1$ and by induction true for $n \geq 1$	**B1**

2.

$u_1 = 1^2(1-1)+1 = 1$	**M1**
(so true for $n=1$. Assume true for $n=k$) $u_{k+1} = k^2(k-1)+1+k(3k+1)$	**M1** **A1**
$= k(k^2-k+3k+1)+1 = k(k+1)^2+1$ which implies is true for $n = k+1$ As result is true for $n = 1$ this implies true for all positive integers and so result is true by induction	**M1** **A1**

Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Proof by Induction (5) Chapter Reference: Core Pure 1, Chapter 8	10 minutes

1. $f(n) = 2^n + 6^n$,

   Show that $f(k + 1) - f(k)$ is divisible by 8(4) **(4)**

2. Prove by induction, that for $n \in Z^+$

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

Solutions

1.

$n = 1$: $f(1) = 2^1 + 6^1 = 8$, **which is divisible** by 8	**B1**
Assume f(k) divisible by 8 and try to	**M1**
use $f(k+1) - f(k)$ or $f(k+1) + f(k)$ including factorising $6^k = 2^k 3^k$ $= 2^3 2^{k-3}(1 + 5.3^k)$ or $= 2^3 2^{k-3}(3 + 7.3^k)$ o.e.	**A1**
Deduction that result is implied for $n = k + 1$ and so is true for positive integers by induction (must include explanation of why $n = 2$ is also true here)	**A1**

2.

$n = 1;\quad \text{LHS} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $\text{RHS} = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ As LHS = RHS, the matrix result is true for $n = 1$.	**B1**
Assume that the matrix equation is true for $n = k$, ie. $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ With $n = k + 1$ the matrix equation becomes $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 9(3^k - 1) + 6 & 0 + 1 \end{pmatrix}$ or $\begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 6.3^k + 3(3^k - 1) & 0 + 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k) - 3 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3(3^k) - 1) & 1 \end{pmatrix}$	**M1** **A1**
$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$	**M1** **A1**
If the result is true for $n = k$,(1) then it is now true for $n = k + 1$. (2) As the result has shown to be true for $n = 1$,(3) then the result is true for all n. (4) **All 4** **aspects need to be mentioned at some point for the** **last A1.**	**A1**