



1. Prove by induction that, for $n \in \mathbb{Z}^+$:

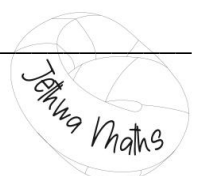
$$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

2. Prove by induction that, for $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

(5)



Solutions

1.

When $n = 1$, LHS = $\frac{1}{1 \times 2} = \frac{1}{2}$, RHS = $\frac{1}{1+1} = \frac{1}{2}$. So LHS = RHS and result true for $n = 1$	B1 M1
Assume true for $n = k$; $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$ and so $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$	M1
$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$	M1 A1
and so result is true for $n = k + 1$ (and by induction true for $n \in \mathbf{Z}^+$)	B1

2.

If $n = 1$, $\sum_{r=1}^n r(r+3) = 1 \times 4 = 4$ and $\frac{1}{3}n(n+1)(n+5) = \frac{1}{3} \times 1 \times 2 \times 6 = 4$,	B1
(so true for $n = 1$. Assume true for $n = k$) So $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$ $= \frac{1}{3}(k+1)[k(k+5) + 3(k+4)] = \frac{1}{3}(k+1)[k^2 + 8k + 12]$	M1
$= \frac{1}{3}(k+1)(k+2)(k+6)$ which implies is true for $n = k + 1$	A1
As result is true for $n = 1$ this implies true for all positive integers and so result is true by induction	M1 A1



Further Maths
A-Level Starter
Activity



Topic: Proof by Induction (2)

Chapter Reference: Core Pure 1, Chapter 8

**10
minutes**

1. Prove by induction, that for $n \in \mathbb{Z}^+$,
 $f(n) = 5^n + 8n + 3$ is divisible by 4.

(7)

2. $f(n) = 2^n + 6^n$,
Show that $f(k+1) = 6f(k) - 4(2^k)$

(3)



Solutions

1.

$f(1) = 5 + 8 + 3 = 16$, (which is divisible by 4). (\therefore True for $n = 1$).	B1
Using the formula to write down $f(k + 1)$, $f(k + 1) = 5^{k+1} + 8(k + 1) + 3$	M1 A1
$f(k + 1) - f(k) = 5^{k+1} + 8(k + 1) + 3 - 5^k - 8k - 3$ $= 5(5^k) + 8k + 8 + 3 - 5^k - 8k - 3 = 4(5^k) + 8$	M1 A1
$f(k + 1) = 4(5^k + 2) + f(k)$, which is divisible by 4	A1
\therefore True for $n = k + 1$ if true for $n = k$. True for $n = 1$, \therefore true for all n .	A1

2. Either LHS:

$LHS = f(k + 1) = 2^{k+1} + 6^{k+1}$	M1
$= 2(2^k) + 6(6^k)$	A1
$= 6(2^k + 6^k) - 4(2^k) = 6f(k) - 4(2^k)$	A1

Either RHS

$= 6f(k) - 4(2^k) = 6(2^k + 6^k) - 4(2^k)$	M1
$= 2(2^k) + 6(6^k)$	A1
$= 2^{k+1} + 6^{k+1} = f(k + 1)$	A1

or,

$f(k + 1) - 6f(k) = 2^{k+1} + 6^{k+1} - 6(2^k + 6^k)$	M1
$= (2 - 6)(2^k) = -4 \cdot 2^k$	A1
and so $f(k + 1) = 6f(k) - 4(2^k)$	A1



Solutions

1.

For $n = 1$, $\begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1$ (\therefore True for $n = 1$.)	B1
$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$	M1 A1 A1
$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$	M1 A1
\therefore True for $n = k + 1$ if true for $n = k$. True for $n = 1$, \therefore true for all n	A1





1. A series of positive integers u_1, u_2, u_3, \dots is defined by

$$u_1 = 6 \text{ and } u_{n+1} = 6u_n - 5 \text{ for } n \geq 1$$

Prove by induction that $u_n = 5 \times 6^{n-1} + 1$ for $n \geq 1$

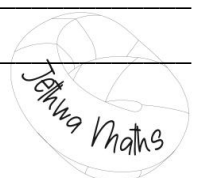
(5)

2. A sequence of positive integers is defined by

$$u_1 = 1 \text{ and } u_{n+1} = u_n + n(3n + 1), \quad n \in \mathbb{Z}^+$$

Prove by induction that $u_n = n^2(n - 1) + 1, \quad n \in \mathbb{Z}^+$

(5)



Solutions

1.

At $n=1$, $u_n = 5 \times 6^0 + 1 = 6$ and so result true for $n = 1$	B1
Assume true for $n = k$; $u_k = 5 \times 6^{k-1} + 1$, and so $u_{k+1} = 6(5 \times 6^{k-1} + 1) - 5$	M1 A1
$\therefore u_{k+1} = 5 \times 6^k + 6 - 5 \quad \therefore u_{k+1} = 5 \times 6^k + 1$	A1
and so result is true for $n = k + 1$ and by induction true for $n \geq 1$	B1

2.

$u_1 = 1^2(1-1) + 1 = 1$	M1
(so true for $n = 1$. Assume true for $n = k$) $u_{k+1} = k^2(k-1) + 1 + k(3k+1)$	M1 A1
$= k(k^2 - k + 3k + 1) + 1 = k(k+1)^2 + 1$ which implies is true for $n = k + 1$ As result is true for $n = 1$ this implies true for all positive integers and so result is true by induction	M1 A1





1. $f(n) = 2^n + 6^n$,

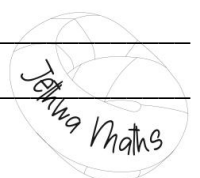
Show that $f(k + 1) - f(k)$ is divisible by 8(4)

(4)

2. Prove by induction, that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)



Solutions

1.

$n = 1: f(1) = 2^1 + 6^1 = 8$, which is divisible by 8	B1
Assume $f(k)$ divisible by 8 and try to	M1
use $f(k+1) - f(k)$ or $f(k+1) + f(k)$ including factorising $6^k = 2^k 3^k$ $= 2^3 2^{k-3} (1 + 5 \cdot 3^k)$ or $= 2^3 2^{k-3} (3 + 7 \cdot 3^k)$ o.e.	A1
Deduction that result is implied for $n = k + 1$ and so is true for positive integers by induction (must include explanation of why $n = 2$ is also true here)	A1

2.

$n = 1; \text{ LHS} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $\text{RHS} = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ <p>As LHS = RHS, the matrix result is true for $n = 1$.</p>	B1
<p>Assume that the matrix equation is true for $n = k$,</p> <p>ie. $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$</p> <hr/> <p>With $n = k + 1$ the matrix equation becomes</p> $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 9(3^k - 1) + 6 & 0 + 1 \end{pmatrix} \text{ or } \begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 6 \cdot 3^k + 3(3^k - 1) & 0 + 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k) - 3 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3(3^k) - 1) & 1 \end{pmatrix}$	M1 A1
$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$	M1 A1
<p>If the result is true for $n = k$ then it is now true for $n = k + 1$. (2) As the result has shown to be true for $n = 1$, (3) then the result is true for all n. (4) All 4 aspects need to be mentioned at some point for the last A1.</p>	A1

