

Further Maths  
A-Level Starter  
Activity



Topic: Modelling with Differential  
Equations (1)  
Chapter Reference: Core Pure 2, Chapter 8

8  
minutes

1. The total value of the sales made by a new company in the first  $t$  years of its existence is denoted by  $\pounds V$ . A model is proposed in which the rate of increase of  $V$  is proportional to the square root of  $V$ . The constant of proportionality is  $k$ .

a. Express the model as a differential equation.

Verify by differential that  $V = (\frac{1}{2}kt + c)^2$ , where  $c$  is an arbitrary constant, satisfies this differential equation. (4)

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b. The value of the company's in its first year is  $\pounds 10000$ , and the total value of the sales in the first two years is  $\pounds 40000$ . Find  $V$  in terms of  $t$ . (4)

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## Solutions

1a.

$\frac{dV}{dt} = k\sqrt{V}$	<b>B1</b>
$V = \left(\frac{1}{2}kt + c\right)^2$	<b>M1</b>
$\frac{dV}{dt} = 2\left(\frac{1}{2}kt + c\right) \cdot \frac{1}{2}k$	
$= k\left(\frac{1}{2}kt + c\right)$	<b>A1</b>
$= k\sqrt{V}$	<b>A1</b>

1b.

$\left(\frac{1}{2}kt + c\right)^2 = 10000 \Rightarrow \frac{1}{2}k + c = 100$	<b>B1</b>
$(k + c)^2 = 40000 \Rightarrow k + c = 200$	<b>B1</b>
$\frac{1}{2}k = 100$	<b>M1</b>
$k = 200 \quad c = 0$	
$V = (100t)^2 = 10000t^2$	<b>A1</b>





1. The motion of a particle is modelled by the differential equation

$$v \frac{dv}{dx} + 4x = 0,$$

where  $x$  is its displacement from a fixed point, and  $v$  is its velocity.

Initially  $x = 1$  and  $v = 4$ .

a. Solve the differential equation to show that  $v^2 = 20 - 4x^2$  **(4)**

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Now consider motion for which  $x = \cos 2t + 2 \sin 2t$ , where  $x$  is the displacement from a fixed point at time  $t$ .

b. Verify that, when  $t = 0$ ,  $x = 1$ . Use the fact that  $v = \frac{dx}{dt}$  to verify that when  $t = 0$ ,  $v = 4$ . **(4)**

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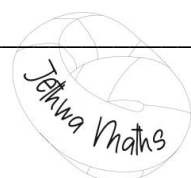
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## Solutions

1a.

$v \frac{dv}{dx} + 4x = 0$ $\int v \, dv = - \int 4x \, dx$ $\frac{1}{2}v^2 = -2x^2 + c$ When $x = 1, v = 4, \text{ so } c = 10$ So $v^2 = 20 - 4x^2$	<b>M1</b> <b>A1</b> <b>B1</b>  <b>A1</b>
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1b.

$x = \cos 2t + 2 \sin 2t$ when $t = 0, x = \cos 0 + 2 \sin 0 = 1$ $v = \frac{dx}{dt} = -2 \sin 2t + 4 \cos 2t$ $v = 4 \cos 0 - 2 \sin 0 = 4$	<b>B1</b> <b>M1</b>  <b>A1</b> <b>A1</b>
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1. The displacement  $x$  at time  $t$  of an oscillating system from a fixed point is given by

$$\ddot{x} + 2\lambda\dot{x} + 5x = 0,$$

where  $\lambda \geq 0$ .

a. For what value of  $\lambda$  is the motion simple harmonic? State the general solution in this case. (3)

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b. Find the range of values of  $\lambda$  for which the system is under-damped. (3)

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Consider the case  $\lambda = 1$ .

c. Find the general solution of the differential equation. (3)

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## Solutions

1a.

$\lambda = 0$ $x = A \cos \sqrt{5}t + B \sin \sqrt{5}t$	<b>B1</b> <b>M1</b> <b>A1</b>
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1b.

$(2\lambda)^2 - 4.5 < 0$ $0 < \lambda < \sqrt{5}$	<b>M1</b> <b>A1</b> <b>A1</b>
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1c.

$m^2 + 2m + 5 = 0$ $m = -1 \pm 2i$ $x = e^{-t}(C \cos 2t + D \sin 2t)$	<b>M1</b> <b>A1</b> <b>F1</b>
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## Solutions

1.

$m^2 + 25 = 0$	<b>M1</b>
$m = \pm 5i$	<b>A1</b>
C.F. $y = A \cos 5t + B \sin 5t$	<b>F1</b>
P.I. $at \cos 5t + bt \sin 5t$	<b>B1</b>
$\dot{y} = a \cos 5t - 5at \sin 5t + b \sin 5t + 5bt \cos 5t$	<b>M1</b>
$\ddot{y} = -10a \sin 5t - 25at \cos 5t + 10b \cos 5t - 25bt \sin 5t$	<b>M1</b>
$\Rightarrow b = 2, a = 0$	<b>A1</b>
P.I. $y = 2t \sin 5t$	
G.S. $y = 2t \sin 5t + A \cos 5t + B \sin 5t$	<b>F1</b>







## Solutions

1.

$\begin{aligned}\ddot{x} &= -5\dot{x} + 4\dot{y} - 2e^{-2t} \\ &= -5\dot{x} + 4(-9x + 7y + 3e^{-2t}) - 2e^{-2t} \\ &= -5\dot{x} - 36 + \frac{28}{4}(\dot{x} + 5x - e^{-2t}) + 10e^{-2t} \\ \therefore \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x &= 3e^{-2t}\end{aligned}$	<b>M1</b> <b>M1</b> <b>M1</b> <b>M1</b> <b>E1</b>
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