Further Maths A-Level Starter Activity

Topic: Modelling with Differential Equations (1)
Chapter Reference: Core Pure 2, Chapter 8

8 minutes

1. The total value of the sales made by a new company in the first t years of its existence is denoted by £V. A model is proposed in which the rate of increase of V is proportional to the square root of V. The constant of proportionality is k.

a. Express the model as a differential equation.

Verify by differential that $V = (\frac{1}{2}kt + c)^2$, where c is an arbitrary constant, satisfies this differential equation. **(4)**

b. The value of the company's in its first year is £10000, and the total value of the sales in the first two years is £40000. Find V in terms of t. **(4)**

Solutions

1a.

$\frac{dV}{dt} = k\sqrt{V}$ $V = (\frac{1}{2}kt + c)^2$ $\frac{dV}{dt} = 2(\frac{1}{2}kt + c)^2.\frac{1}{2}k$ $= k(1/2\,kt + c)$ $= k\sqrt{V}$	**B1** **M1** **A1** **A1**

1b.

$(\frac{1}{2}kt + c)^2 = 10000 \Rightarrow 1/2\,k + c = 100$ $(k + c)^2 = 40000 \Rightarrow k + c = 200$ $1/2\,k = 100$ $k = 200 \quad c = 0$ $V = (100t)^2 = 10000t^2$	**B1** **B1** **M1** **A1**

Further Maths A-Level Starter Activity

Topic: Modelling with Differential Equations (2) Chapter Reference: Core Pure 2, Chapter 8	8 minutes

1. The motion of a particle is modelled by the differential equation

$$v\frac{dv}{dx} + 4x = 0,$$

where x is its displacement from a fixed point, and v is its velocity.

Initially $x = 1$ and $v = 4$.

a. Solve the differential equation to show that $v^2 = 20 - 4x^2$ **(4)**

Now consider motion for which $x = \cos 2t + 2\sin 2t$, where x is the displacement from a fixed point at time t.

b. Verify that, when $t = 0$, $x = 1$. Use the fact that $v = \frac{dx}{dt}$ to verify that when $t = 0$, $v = 4$. **(4)**

Solutions

1a.

$v\,{}^{dv}/_{dx} + 4x = 0$ $\int v\,dv = -\int 4x\,dx$ $\frac{1}{2}v^2 = -2x^2 + c$ When $x = 1, v = 4, so\ c = 10$ So $v^2 = 20 - 4x^2$	**M1** **A1** **B1** **A1**

1b.

$x = \cos 2t + 2\sin 2t$	**B1**
when $t = 0,\ x = \cos 0 + 2\sin 0 = 1$	**M1**
$v = \frac{dx}{dt} = -2\sin 2t + 4\cos 2t$	**A1**
$v = 4\cos 0 - 2\sin 0 = 4$	**A1**

Further Maths A-Level Starter Activity

Topic: Modelling with Differential Equations (3)

Chapter Reference: Core Pure 2, Chapter 8

9 minutes

1. The displacement x at time t of an oscillating system from a fixed point is given by

$$\ddot{x} + 2\lambda\dot{x} + 5x = 0,$$

where $\lambda \geq 0$.

a. For what value of λ is the motion simple harmonic? State the general solution in this case. **(3)**

b. Find the range of values of λ for which the system is under-damped. **(3)**

Consider the case λ = 1.

c. Find the general solution of the differential equation. **(3)**

Solutions

1a.

$\lambda = 0$ $x = A\cos\sqrt{5}t + B\sin\sqrt{5}t$	**B1** **M1** **A1**

1b.

$(2\lambda)^2 - 4.5 < 0$ $0 < \lambda < \sqrt{5}$	**M1** **A1** **A1**

1c.

$m^2 + 2m + 5 = 0$ $m = -1 \pm 2i$ $x = e^{-t}(C\cos 2t + D\sin 2t)$	**M1** **A1** **F1**

1. A car travels over a rough surface. The vertical motion of the front suspension is modelled by the differential equation

$$\frac{d^2y}{dt^2} + 25y = 20\cos 5t,$$

where y is the vertical displacement of the top of the suspension and t is time.

Find the general solution. **(8)**

Solutions

1.

$m^2 + 25 = 0$	**M1**
$m = \pm 5i$	**A1**
C.F. $y = A\cos 5t + B\sin 5t$	**F1**
P.I. $at\cos 5t + bt\sin 5t$	**B1**
$\dot{y} = \text{a}\cos 5t - 5at\sin 5t + b\sin 5t + 5bt\cos 5t$	**M1**
$\ddot{y} = -10\,\text{a}\sin 5t - 25at\cos 5t + 10b\cos 5t - 25bt\sin 5t$	**M1**
$\Rightarrow$ b = 2, a = 0	**A1**
P.I. $y = 2t\sin 5t$	
G.S. $y = 2t\sin 5t + A\cos 5t + B\sin 5t$	**F1**

Topic: Modelling with Differential Equations (5) Chapter Reference: Core Pure 2, Chapter 8	5 minutes

1. The following simultaneous differential equations are to be solved.

$$\frac{dx}{dt} = -5x + 4y + e^{-2t},$$

$$\frac{dy}{dt} = -9x + 7y + 3e^{-2t}$$

Show that $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 3e^{-2t}$ **(5)**

Solutions

1.

$\ddot{x} = -5\dot{x} + 4\dot{y} - 2e^{-2t}$ $= -5\dot{x} + 4(-9x + 7y + 3e^{-2t}) - 2e^{-2t}$ $= -5\dot{x} - 36 + \frac{28}{4}(\dot{x} + 5x - e^{-2t}) + 10e^{-2t}$ $\therefore \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 3e^{-2t}$	**M1** **M1** **M1** **M1** **E1**