Further Maths A-Level Starter Activity

Jethwa Maths

Topic: Probability Generating Functions (1) Chapter Reference: Further Statistics 1, Chapter 7	9 minutes

1. The probability generating function of a discrete random variable Y is given by

$$\mathrm{G}_Y(t) = k(1 + 2t + 3t^2)^2$$

a. Find the value of k. **(2)**

b. Find $\mathrm{P}(Y = 2)$ **(2)**

2. The probability generating function of a discrete random variable X is given by

$$\mathrm{G}_X(t) = k(1 + t + 3t^2)^2$$

a. Find the value of k. **(2)**

b. Write down the probability distribution of X. **(3)**

Solutions

1a.

$G_Y(1) = 1$ $k(1 + 2 + 3)^2 = 1$ $36k = 1$	**M1**
$k = \frac{1}{36}$ (oe)	**A1**

1b.

$G_Y(t) = \frac{1}{36}(1 + 2t + 3t^2)^2$ $G_Y(t) = \frac{1}{36}(1 + 4t + 10t^2 + 12t^3 + 9t^4)$	**M1**
$P(Y = 2) = \frac{1}{36} \times 10 = \frac{10}{36} = \frac{5}{18}$	**A1**

2a.

$G_X(1) = 1$ $k(1 + 1 + 3)^2 = 1$ $25k = 1$	**M1**
$k = \frac{1}{25}$ (oe)	**A1**

2b.

$G_X(t) = \frac{1}{25}(1 + t + 3t^2)^2$ $G_X(t) = \frac{1}{25}(1 + 2t + 7t^2 + 6t^3 + 9t^4)$	**M1**
$G_X(t) = \frac{1}{25} + \frac{2}{25}t + \frac{7}{25}t^2 + \frac{6}{25}t^3 + \frac{9}{25}t^4$	**M1**
(table below)	**A1**

x	0	1	2	3	4
P(*X* = *x*)	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{7}{25}$	$\frac{6}{25}$	$\frac{9}{25}$

Further Maths A-Level Starter Activity

Topic: Probability Generating Functions of Standard distributions (2)
Chapter Reference: Further Statistics 1, Chapter 7

9 minutes

1. Write down the probability generating functions for the following distributions:

a. $X \sim \mathrm{B}(5, 0.2)$ **(2)**

b. $X \sim \mathrm{Po}(1.5)$ **(1)**

c. $X \sim \mathrm{Geo}(0.9)$ **(2)**

d. $X \sim$ Negative $\mathrm{B}(6, 0.2)$ **(2)**

2. A tailor notices that the flaws in a roll of cloth occur at an average rate of 0.7 per metre.

a. Suggest a suitable model for the random variable X, the number of flaws in a metre of cloth. **(1)**

b. Write down the probability generating function of X. **(1)**

Solutions

1a.

$G_X(t) = (1 - p + pt)^n$ $= (1 - 0.2 + 0.2t)^5$ $= (0.8 + 0.2t)^5$ $= 0.2^5(4 + t)^5$	**M1**
$= \frac{1}{3125}(4 + t)^5$	**A1**

1b.

$G_X(t) = e^{\lambda(t-1)}$ $= e^{1.5(t-1)}$	**A1**

1c.

$G_X(t) = \frac{pt}{1-(1-p)t}$ $= \frac{0.9t}{1-(1-0.9)t}$ $= \frac{0.9t}{1-0.1t}$	**M1**
$= \frac{9t}{10-t}$	**A1**

1d.

$G_X(t) = \left(\frac{pt}{1-(1-p)t}\right)^r$ $= \left(\frac{0.2t}{1-(1-0.2)t}\right)^6$ $= \left(\frac{0.2t}{1-0.8t}\right)^6$	**M1**
$= \left(\frac{2t}{10-8t}\right)^6 = \left(\frac{t}{5-4t}\right)^6$	**A1**

2a.

$X \sim \text{Po}(0.7)$	**A1**

2b.

$G_X(t) = e^{\lambda(t-1)}$ $= e^{0.7(t-1)}$	**A1**

Further Maths A-Level Starter Activity

Topic: Probability Generating Functions with Mean and Variance (3)

Chapter Reference: Further Statistics 1, Chapter 7

6 minutes

1. A discrete random variable Y has probability generating function $G_Y(t) = \frac{10}{(7-t)^2}$.

a. Find the mean and variance of Y. **(5)**

b. Find the standard deviation of Y. **(1)**

Solutions

1a.

$G_Y(t) = \frac{10}{(7-t)^2} = 10(7-t)^{-2}$ $G'_Y(t) = -1\times -2\times 10(7-t)^{-3} = \frac{20}{(7-t)^3}$	**M1**
$E(Y) = G'_Y(1) = \frac{5}{54}$	**A1**
$G''_Y(t) = -1\times -3\times 20(7-t)^{-4} = \frac{60}{(7-t)^4}$	**M1**
$G''_Y(1) = \frac{5}{108}$	**M1**
Hence $\text{Var}(Y) = G''_Y(1) + G'_Y(1) + (G'_Y(1))^2$ $= \frac{5}{108} + \frac{5}{54} - \frac{25}{2916}$ $= \frac{95}{729}$	**A1**

1b.

Standard deviation of $Y = \sqrt{\text{Var}(Y)} = \sqrt{\frac{95}{729}} = \frac{\sqrt{95}}{27} = 0.361$ (3.d.p) (oe)	**A1**

Further Maths A-Level Starter Activity

Topic: Probability Generating Functions (4)

Chapter Reference: Further Statistics 1, Chapter 7

9 minutes

1. The discrete random variable Y has probability generating function

$$G_Y(t) = 0.09t^2 + 0.24t^3 + 0.34t^4 + 0.24t^5 + 0.09t^6$$

a. Find the mean and variance of Y. **(5)**

Y is the sum of two independent observations of a random variable X.

b. Find the probability generating function X, expressing your answer as a cubic polynomial in t. **(3)**

c. Write down the value of $P(X = 2)$. **(1)**

Solutions

1a.

$G'_Y(t) = 0.18t + 0.72t^2 + 1.36t^3 + 1.2t^4 + 0.54t^5$ ≥ 3 terms correct	**M1**
Sub t =1, (0.18 + 0.72 + 1.36 + 1.2 + 0.54)	**M1**
= 4 (E(Y))	**A1**
$G''_Y(t) = 0.18 + 1.44t + 4.08t^2 + 4.8t^3 + 2.7t^4$	**M1**
Sub t=1, 0.18 + 1.44 + 4.08 + 4.8 + 2.7 = 13.2 Use correct formula for Var, 13.2 + 4 - 16	**M1**
= 1.2	**A1**

1b.

Attempt to factorise into $(0.3t + \ldots)(0.3t + \ldots)$ Attempt to find $\sqrt{(G_Y(t))}$ seen or implied.	**M1M1**
$= 0.3t + 0.4t^2 + 0.3t^3$	**A1**

1c.

0.4	**A1**

Further Maths A-Level Starter Activity

Topic: Probability Generating Functions (5)

Chapter Reference: Further Statistics 1, Chapter 7

7 minutes

1. The probability generating function of a discrete random variable X and Y is given by

$$G_X(t) = \frac{1}{2}(t + t^2) \qquad G_Y(t) = \frac{2}{3}(1 + 2t + 3t^2)^2$$

Given that X and Y are independent, find the probability generating function of $Z = X + Y$ **(2)**

2. A random variable X has probability generating function $G_X(t) = \frac{1}{4} + \frac{1}{2}t^2 + t^4$.

a. Find $E(X)$. **(3)**

A random variable $Y = 2X - 1$.

b. Find the probability generating function of Y. **(2)**

<u>Solutions</u>

1.

$G_Z(t) = G_X(t)G_Y(t)$ $= \frac{1}{2}(t + t^2) \times \frac{2}{3}(1 + 2t + 3t^2)^2$	**M1**
$= \frac{1}{3}(t + t^2)(1 + 2t + 3t^2)^2$	**A1**

2a.

$G'_X(t) = t + 4t^3$	**M1**
$G_X(1) = 1 + 4$	**M1**
$= 5$	**A1**

2b.

$G_Y(t) = t^{-1}(G_X(t^2))$ $= t^{-1}(\frac{1}{4} + \frac{1}{2}t^4 + t^8)$	**M1**
$= \frac{1}{4}t^{-1} + \frac{1}{2}t^3 + t^7$	**A1**