Topic: Hyperbolic functions (1)

Chapter Reference: Core Pure 2, Chapter 6

6 minutes

1. A function is given by,

$$f(x) = 5\cosh x - 4\sinh x, \qquad x \in \mathbb{R}$$

Show that $f(x) = \frac{1}{2}(e^x + 9e^{-x})$ **(2)**

2. Using the definitions of $\cosh x$ and $\sinh x$ n terms of e^x and e^{-x}, prove that

$$\sinh 2x = 2\sinh x \cosh x.$$ **(4)**

Solutions

1.

$f(x) = 5\cosh x - 4\sinh x = 5 \times {}^{1}\!/_{2}\,(e^x + e^{-x}) - 4 \times {}^{1}\!/_{2}\,(e^x - e^{-x})$ $= \frac{1}{2}(e^x + 9e^{-x})$	**M1** **A1**

2.

Correct def$''$ of $\cosh x$ and $\sinh x$ Expand $2 \cdot \frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^x + e^{-x})$ $\frac{1}{2}(e^{2x} - e^{-2x})$	**B1** **B1** **M1** **A1**

Further Maths A-Level Starter Activity

Topic: Hyperbolic functions (2)	8 minutes
Chapter Reference: Core Pure 2, Chapter 6	

1. $f(x) = \text{artanh}\, x\ , x \in \mathbb{R}, -1 < x < 1.$

a. Show clearly that

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) x \in \mathbb{R}, -1 < x < 1.$$

(5)

b. Hence simplify fully

$$g(x) = \text{artanh}\left(\frac{x^2-1}{x^2+1}\right), x > 0 \cdot$$

(3)

Solutions

1a.

$y = artanhx$ $tanhy = x$ $\dfrac{e^{2y}-1}{e^{2y}+1} = x$ $e^{2y}-1 = xe^{2y}+x$ $e^{2y} = \dfrac{1+x}{1-x}$ $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ $\therefore\ artanhx = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	**M1** **M1** **A1** **A1** **A1**

1b.

$g(x) = \dfrac{1}{2}\ln\left[\dfrac{1+\dfrac{x^2-1}{x^2+1}}{1-\dfrac{x^2-1}{x^2+1}}\right]$ $g(x) = \ln x$	**M1** **M1** **A1**

1. Given that,

$$2\cosh^2 x - 1 \equiv \cosh 2x.$$

Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials. **(3)**

2. Solve the hyperbolic equation,

$$4 + 6(e^{2x} + 1)\tanh x = 11\cosh x + 11\sinh x.$$

(4)

Solutions

1.

LHS = $2\cosh^2 x = 2\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)^2 - 1$ $= \frac{1}{2}(e^{2x}+e^{-2x})$ = $\cosh 2x$ = RHS	**M1** **A1** **A1**

2.

$4+6(e^{2x}+1)\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\frac{11}{2}e^{x}+\frac{11}{2}e^{-x}+\frac{11}{2}e^{x}-\frac{11}{2}e^{-x}$ $\Longrightarrow\ 6e^{2x}-11e^{x}-2=0$ $\Rightarrow e^{x}=2$ $\therefore x=\ln 2$	**M1** **M1** **A1** **A1**

Topic: Hyperbolic functions (4)

Chapter Reference: Core Pure 2, Chapter 6

5 minutes

1. Given that,

$$f(x) = arsinhx + arsinh\left(\frac{1}{x}\right), x \in \mathbb{R}, x \neq 0.$$

Show clearly that $f'(x) = \frac{x^2 - |x|}{x^2\sqrt{x^{2+1}}}$. **(5)**

Solutions

1.

$f(x) = arsinhx + arsinh\left(\frac{1}{x}\right)$ $f'(x) = \frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{1+\frac{1}{x^2}}} \times \left(\frac{-1}{x^2}\right)$ $f'(x) = \frac{x^2 - \lvert x \rvert}{x\sqrt[2]{x^{2+1}}}$	**M1** **M1** **A1** **A1** **A1**

Further Maths A-Level Starter Activity

Topic: Hyperbolic functions (5)

Chapter Reference: Core Pure 2, Chapter 6

8 minutes

1. If $0 < k < \sqrt{2} - 1$ prove that

$$\int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1}\, dx = \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x}\, dx.$$

You need not evaluate these integrals. **(8)**

Solutions

1.

LHS $\int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1}\,dx = \int_k^{\frac{1-k}{1+k}} \ln x \frac{1}{x^2-1}\,dx$ $= \left[-\ln(x)(\operatorname{artanh} x)\right]_k^{\frac{1-k}{1+k}} - \int_k^{\frac{1-k}{1+k}} \frac{-1}{x} \operatorname{artanh} x \; dx$ $= \left[\ln(x)(\operatorname{artanh} x)\right]_{\frac{1-k}{1+k}}^{k} + \int_k^{\frac{1-k}{1+k}} \frac{1}{x} \operatorname{artanh} x \; dx$ Show that $\left[\ln(x)(\operatorname{artanh} x)\right]_{\frac{1-k}{1+k}}^{k} = 0$ $\therefore \int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1}\,dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x}\,dx$	**M1** **M1** **M1** **A1** **A1** **B1** **M1** **A1**