A-Level Unit Test: Algebra and Functions

Composite Functions

1. $g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}$, $x > 3$

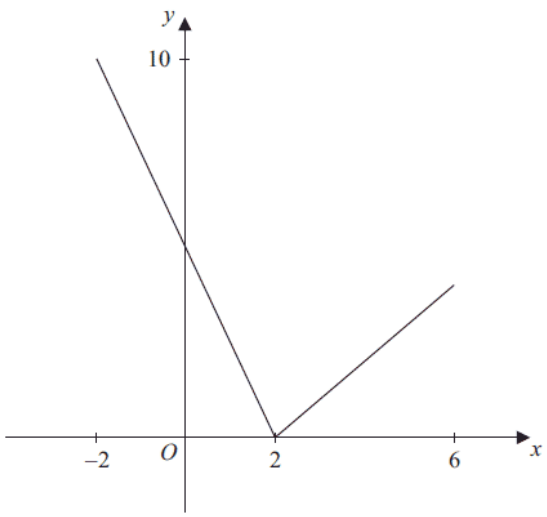
a. Show that $(x) = \frac{x+1}{x-2}$, $x > 3$ **(4)**

b. Find the range of g **(2)**

c. Find the exact value of a for which $g(a) = g^{-1}(a)$ **(4)**

2. The function f has domain $-2 \le x \le 6$ and is linear from (-2, 10) to (2, 0) and from (2, 0) to (6, 4). A sketch of the graph $y = f(x)$ is shown in figure 1.

Figure 1



a. Write down the range of f

b. Find ff(0) **(1)**

A function g is defined by **(2)**

$$g : x \to \frac{4+3x}{5-x}, \quad x \in \mathbb{R}, \quad x \neq 5$$

c. Find $g^{-1}(x)$ **(3)**

d. Solve the equation $gf(x) = 16$ **(5)**

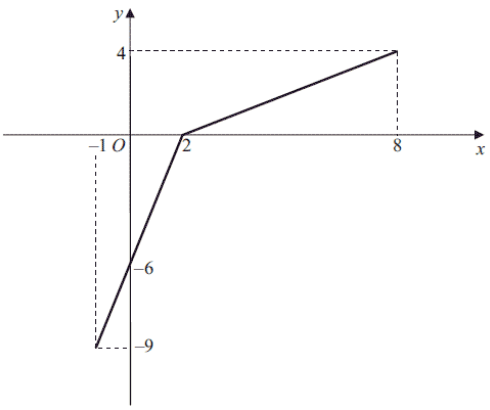
3. The function f is defined by

$$f : x \to \frac{3-2x}{x-5}, x \in \mathbb{R}, \quad x \neq 5$$

a. Find $f^{-1}(x)$. **(3)**

The function g has domain $-1 \le x \le 8$ and is linear from (-1, -9) to (2, 0) and from (2, 0) to (8, 4). Figure 2 shows a sketch of the graph $y = g(x)$

Figure 2



b. Write down the range of g. **(1)**

c. Find gg(2) **(2)**

d. Find fg(8) **(2)**

e. On the same diagram sketch the graphs $y = g(x)$ and $y = g^{-1}(x)$. Show the coordinates of each point at which the graph meets or cuts the aces. **(3)**

f. State the domain of the inverse function g^{-1}. **(1)**

4. The function f is defined by

$$f : x \to \frac{3x-5}{x+1}, x \in \mathbb{R},\ x \neq -1$$

a. Find $f^{-1}(x)$ **(3)**

b. Show that $ff(x) = \frac{x+1}{x-1}$ $x \in \mathbb{R},\ x \neq -1$, where a is an integer to be found. **(4)**

The function g is defined by

$$g : x \to x^2 - 3x, x \in \mathbb{R}, 0 \leq x \leq 5$$

c. Find the value of fg(2) **(2)**

d. Find the range of g **(3)**

e. Explain why the function g does not have an inverse **(1)**

5. The functions f and g are defined by

$$f : x \to 2x + \ln x, x \in \mathbb{R}$$
$$g : x \to e^{2x}, x \in \mathbb{R}$$

a. Prove that the composite function gf is: **(4)**

$$gf : x \to 4e^{4x}, x \in \mathbb{R}$$

b. Sketch the curve with equation $y = gf(x)$, and show the coordinates of the point where the curve cuts the y-axis. **(3)**

c. Write down the range of gf **(1)**

d. Find the value of x for which $\frac{d}{dx}[gf(x)] = 3$, giving your answer to 3 significant figures. **(4)**

6. A function g is defined by

$$g(x) = 3 + \sqrt{x+2}\ , x \geq -2$$

a. State the range of g **(1)**

b. Find $g^{-1}(x)$ and state its domain. **(3)**

c. Find the exact value of x for which, $g(x) = x$ **(4)**

d. Hence state the value of a for which $g(a) = g^{-1}(a)$ **(1)**

7. The functions f and g are defined by:

$$f : x \to 2x + 3, x \in \mathbb{R}$$
$$g : x \to 3 - 4x, x \in \mathbb{R}$$

a. State the range of f **(2)**

b. Find fg(1) **(2)**

c. Find g^{-1}, the inverse function of g. **(2)**

d. Solve the equation **(5)**

$$gg(x) + [g(x))]^2 = 0$$

8. The functions f and g are defined by:

$$f : x \to 3x + \ln x\ , x > 0, x \in \mathbb{R}$$
$$g : x \to e^{x^2},\ x \in \mathbb{R}$$

a. Write down the range of g **(1)**

b. Show that the composite function fg is defined by **(2)**

$$fg : \to x^2 + 3e^{x^2}, x \in \mathbb{R}$$

c. Write down the range of fg. **(1)**

Total marks: 82

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Mark Scheme

1a.

$\frac{x}{x+3}+\frac{3(2x+1)}{x^2+x-6}$ $=\frac{x}{x+3}+\frac{3(2x+1)}{(x+3)(x-2)}$	**M1**
$=\frac{x(x-2)+3(2x+1)}{(x+3)(x-2)}$	**M1**
$=\frac{x^2-2x+6x+3}{(x+3)(x-2)}$ $=\frac{x^2+4x+3}{(x+3)(x-2)}$	**M1**
$=\frac{(x+3)(x+1)}{(x+3)(x-2)}$ $=\frac{(x+1)}{(x-2)}$	**M1**

1b.

$x > 3$, $g(x) = \frac{(3+1)}{(3-2)} = 4$	**M1**
$g(x) = 1 \rightarrow$ asymptote Range, $1 < g(x) \leq 4$	**M1**

1c.

$g(x) = \frac{x+1}{x-2} \rightarrow y = \frac{x+1}{x-2}$ $x = \frac{y+1}{y-2}$ $x(y-2) = y + 1$ $xy - 2x - y = 1$	**M1**
$y(x-1) = 1 + 2x$ $y = \frac{1+2x}{x-1} = g^{-1}(x)$	**M1**
$g(a) = g^{-1}(a)$ $\frac{a+1}{a-2} = \frac{1+2a}{a-1}$ $(a+1)(a-1) = (1+2a)(a-2)$ $a^2 - 1 = 2a^2 - 2 - 3a$	**M1**
$a^2 - 3a - 1 = 0$ $a = \frac{3 \pm \sqrt{13}}{2}$ As $x = a > 3$, $a = \frac{3+\sqrt{13}}{2}$	**M1**

2a.

$0 \leq f(x) \leq 10$	**M1**

2b.

ff(0) = f(5) (using symmetry of line)	**M1**
f(5) = ¾ way across interval, therefore ff(0) = 3.	**M1**

2c.

$y = \frac{4+3x}{5-x} \rightarrow x = \frac{4+3y}{5-y}$ $x(5-y) = 4 + 3y$ $5x - xy = 4 + 3y$	**M1**
$3y + xy = 5x - 4$ $y(3+x) = 5x - 4$	**M1**
$y = \frac{5x-4}{3+x} \rightarrow g^{-1}(x) = \frac{5x-4}{3+x}$	**M1**

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2d.

$g^{-1}gf(x) = g^{-1}(16)$	**M1**
$f(x) = g^{-1}(16) = \frac{5(16)-4}{3+(16)} = 4$	**M1**
Using graph, $f(x) = 4$	**M1**
$x = 6$ or $x = \frac{1}{2}(2) = \frac{2}{5}$ $gf(x) = 16$ when $x = 6$ or $\frac{2}{5}$	**M1 M1**

3a.

$f(x) = \frac{3-2x}{x-5} \rightarrow x = \frac{3-2y}{y-5}$	**M1**
$x(y-5) = 3-2y$ $xy - 5x = 3 - 2y$ $xy + 2y = 3 + 5x$	**M1**
$y(x+2) = 3 + 5x$ $y = \frac{3+5x}{x+2} \rightarrow f^{-1}(x) = \frac{3+5x}{x+2}$	**M1**

3b.

$-9 \leq f(x) \leq 4$	**M1**

3c.

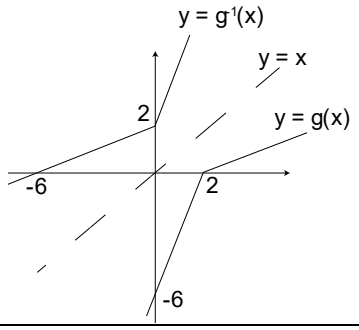
$g[g(2)] = g(0)$	**M1**
$= -6$	**M1**

3d.

$f[g(8)] = f(4)$	**M1**
$f(4) = \frac{3-2(4)}{4-5} = 5$	**M1**

3e.

Attempted reflection in y = x **M1**
$g(x)$ and $g^{-1}(x)$ drawn correctly **M1**
All points cutting axes labelled **M1**



3f.

Domain of $g^{-1}(x) = -9 \leq x \leq 4$	**M1**

4a.

$y = \frac{3x-5}{x+1} \rightarrow x = \frac{3y-5}{y+1}$	**M1**
$x(y+1) = 3y - 5$ $xy + x = 3y - 5$ $3y - xy = x + 5$	**M1**
$y(3-x) = x + 5$ $y = \frac{x+5}{3-x} \rightarrow f^{-1}(x) = \frac{x+5}{3-x}$	**M1**

4b.

$f(x) = \frac{3x-5}{x+1}$	**M1**

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ff(x) = $\frac{3(\frac{3x-5}{x+1})-5}{(\frac{3x-5}{x+1})+1}$	**M1**
ff(x) = $\frac{(\frac{9x-15-5(x+1)}{x+1})}{(\frac{3x-5+x+1}{x+1})} = \frac{4x-20}{4x-4} = \frac{x-5}{x-1}$ a = -5	**M1**

4c.

g(2) = $2^2 - 3(2) = -2$	**M1**
f(-2) = $\frac{3(-2)-5}{(-2)+1} = 11$	**M1**

4d.

g(x) = $x^2 - 3x$ g(0) = 0 g(5) = $5^2 - 3(5) = 25 - 15 = 10$	**M1**
$x^2 - 3x = (x - 1.5)^2 - 2.25$ Therefore minimum point of curve = (1.5, -2.25)	**M1**
Range is: $-2.25 \leq g(x) \leq 10$	**M1**

4e.

The function g cannot have an inverse as the function is not one to one and hence the inverse would be one to many, which is not a function.	**M1**

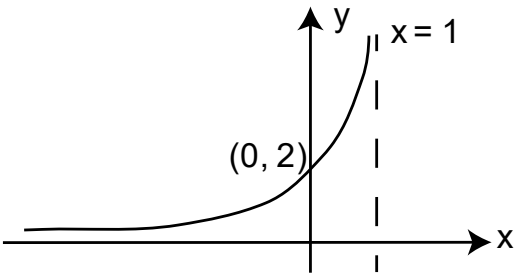
5a.

gf (x) = $e^{2(2x + \ln x)}$ $= e^{4x + 2 \ln x}$	**M1**
$= e^{4x} \times e^{2\ln x}$	**M1**
$= e^{4x} \times 2$	**M1**
$= 2e^{4x}$	**M1**

5b.

When $x = 0$, $y = 2e^0 = 2$ **M1**

Asymptotes at $x = 1$, $y =$ **0 M1 M1**



5c.

Range: fg(x) > 0	**M1**

5d.

$\frac{d}{dx}$gf (x) = 2 x e^{4x} x 4 = $8e^{4x}$	**M1 M1**
$8e^{4x} = 3$ $e^{4x} = \frac{3}{8}$	**M1**
$4x = \ln\frac{3}{8}$ $x = \frac{1}{4} \ln\frac{3}{8}$	**M1**

6a.

Range: $g(x) \geq 3$	**M1**

6b.

$g(x) = 3 + \sqrt{x+2} \rightarrow x = 3 + \sqrt{y+2}$	**M1**
$y = g^{-1}(x) = (x-3)^2 - 2$	**M1**
Domain: $x \geq 3$	**M1**

6c.

$3 + \sqrt{x+2} = x$ $\sqrt{x+2} = x - 3$	**M1**
$x + 2 = (x-3)^2$ $x^2 - 6x + 9 - x - 2 = 0$ $x^2 - 7x - 7 = 0$ $x = \frac{7 \pm \sqrt{21}}{2}$	**M1**
$x \geq 3$, therefore $x = \frac{7+\sqrt{21}}{2}$	**M1**

6d.

A function and it's inverse will be equal therefore, $g(a) = g^{-1}(a)$ $a = \frac{7+\sqrt{21}}{2}$	**M1**

7a.

$f(x) \in \mathbb{R}$	**M1**

7b.

$g(1) = 3 - (4)(1) = -1$	**M1**
$f(-1) = 2(-1) + 3 = 1$	**M1**

7c.

$g(x) = 3 - 4x \rightarrow x = 3 - 4y$	**M1**
$4y = 3 - x$ $y = \frac{1}{4}(3 - x)$	**M1**

7d.

$gg(x) = 3 - 4(3 - 4x) = 3 - 12 + 16x = 16x - 9$	**M1**
$[g(x)]^2 = (3 - 4x)^2 = (3 - 4x)(3 - 4x) = 9 + 16x^2 - 24x$	**M1**
$16x - 9 + 9 + 16x^2 - 24x = 0$ $16x^2 - 8x = 0$	**M1**
$8x(2x - 1) = 0$	**M1**
$x = 0$ $x = \frac{1}{2}$	**M1**

8a.

$g(x) \geq 1$	**M1**

8b.

$3(e^{x^2}) + \ln(e^{x^2})$	**M1**
$= 3e^{x^2} + x^2$	**M1**

8c.

Range: $fg(x) \geq 3$	**M1**

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